Error Bound Derivation

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Let $e^j(\mu) = u_h^j(\mu) - u_{h,N}^j(\mu)$. Then, for $j = 1, \ldots, J$,

$$\left(\frac{e^j - e^{j-1}}{\Delta t}, v\right) + a(e^j, v; \mu) = R_{h,N}^j(v; \mu), \quad \forall\, v \in X_h \ , \tag{1}$$

where the "$g(t^j)\, f(v)$ part" of $R_{h,N}^j$ comes from the u_h^j in e^j and the "rest" of $R_{h,N}^j$ comes from the $u_{h,N}^j$ in e^j. Recall that here $m(w, v; \mu) = (w, v)$, where $(\,\cdot\,,\,\cdot\,)$ is the $L^2(\Omega)$ inner product. Also $e^0 = u_h^0 - u_{h,N}^0 = 0$.

Now choose $v = e^j$ to obtain

$$\left(\frac{e^j - e^{j-1}}{\Delta t}, e^j\right) + a(e^j, e^j; \mu) = R_{h,N}^j(e^j; \mu) \ , \tag{2}$$

or

$$\frac{1}{\Delta t}\,(e^j, e^j) + a(e^j, e^j; \mu) = \frac{1}{\Delta t}\,(e^{j-1}, e^j) + R_{h,N}^j(e^j; \mu) \ . \tag{3}$$

Now by Cauchy-Schwarz and Young,

$$(e^{j-1}, e^j) \le \|e^{j-1}\|\, \|e^j\| \le \frac{1}{2}\, \|e^{j-1}\|^2 + \frac{1}{2}\, \|e^j\|^2 \tag{4}$$

and

$$\begin{aligned} R_{h,N}^j(e^j, \mu) &= (\mathcal{R}^j, e^j)_X \\ &\le \|\mathcal{R}^j\|_X\, \|e^j\|_X \ \le\ \frac{1}{2\alpha_h^{\mathrm{LB}}(\mu)}\, \|\mathcal{R}^j\|_X^2 + \frac{\alpha_h^{\mathrm{LB}}(\mu)}{2}\, \|e^j\|_X^2 \ , \end{aligned} \tag{5}$$

so

$$\frac{1}{2\Delta t}\left(\|e^j\|^2 - \|e^{j-1}\|^2\right) + a(e^j, e^j; \mu) - \frac{\alpha_h^{\mathrm{LB}}(\mu)}{2}\, \|e^j\|_X^2 \le \frac{1}{2\alpha_h^{\mathrm{LB}}(\mu)}\, \delta_{h,N}^2(t^j; \mu) \tag{6}$$

for $j = 1, \ldots, J$. Recall that $\delta_{h,N}(t^j; \mu) \equiv \|\mathcal{R}^j\|_X$.

Now note that $\alpha_h^{\mathrm{LB}}(\mu)\, \|e^j\|_X^2 \le a(e^j, e^j; \mu)$ and hence $-\alpha_h^{\mathrm{LB}}(\mu)\, \|e^j\|_X^2 \ge -a(e^j, e^j; \mu)$ such that

$$a(e^j, e^j; \mu) - \frac{\alpha_h^{\mathrm{LB}}(\mu)}{2}\, \|e^j\|_X^2 \ge \frac{1}{2}\, a(e^j, e^j; \mu) \ge 0 \ . \tag{7}$$

Thus (changing dummy variable j to j')

$$\|e^{j'}\|^2 - \|e^{j'-1}\|^2 \leq \frac{\Delta t}{\alpha_h^{\mathrm{LB}}(\mu)}\, \delta^2_{h,N}(t^{j'},\mu) \tag{8}$$

for $j' = 1, \ldots, J$.

We now sum over j' ("telescope") from $j' = 1$ to j (for $j \leq J$) and use $\|e^0\| = 0$ to obtain

$$\|e^j(\mu)\|^2 \leq \frac{1}{\alpha_h^{\mathrm{LB}}(\mu)}\, \Delta t \sum_{j'=1}^{j} \delta^2_{h,N}(t^{j'};\mu) \tag{9}$$

or

$$\|e^j(\mu)\| \leq \left(\alpha_h^{\mathrm{LB}}(\mu)^{-1} \Delta t \sum_{j'=1}^{j} \delta^2_{h,N}(t^{j'};\mu) \right)^{1/2} \equiv \Delta^{L^2}_{h,N}(t^j;\mu) \tag{10}$$

for $j = 1, \ldots, J$.

We now turn to the output. Since $\ell \in L^2(\Omega)$ we easily obtain

$$\begin{aligned} |s_h^j(\mu) - s_{h,N}^j(\mu)| &= |\ell(u_h^j(\mu)) - \ell(u_{h,N}^j(\mu))| \\ &= |\ell(e^j(\mu))| \\ &\leq \sup_{v \in X_h} \left(\frac{\ell(v)}{\|v\|} \right) \|e^j(\mu)\| \\ &\leq \sup_{v \in X_h} \left(\frac{\ell(v)}{\|v\|} \right) \Delta^{L^2}_{h,N}(t^{j'};\mu) \equiv \Delta^s_{h,N}(t^j;\mu)\ , \end{aligned} \tag{11}$$

which concludes the proof. So there.