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AN IMPROVED ERROR BOUND FOR REDUCED BASIS APPROXIMATION OF LINEAR PARABOLIC PROBLEMS

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Abstract. We consider a space-time variational formulation for linear parabolic partial differential equations. We introduce an associated Petrov-Galerkin truth finite element discretization with favorable discrete inf-sup constant β_δ, the inverse of which enters into error estimates: β_δ is unity for the heat equation; β_δ decreases only linearly in time for non-coercive (but asymptotically stable) convection operators. The latter in turn permits effective long-time *a posteriori* error bounds for reduced basis approximations, in sharp contrast to classical (pessimistic) exponentially growing energy estimates. The paper contains a full analysis and various extensions for the formulation introduced briefly in [13] as well as numerical results for a model reaction-convection-diffusion equation.

1. Introduction

The certified reduced basis method (RBM) has been successfully applied to parabolic equations in the case in which the spatial operator is coercive [3, 4]. However, for problems — linear or nonlinear [7] — in which the spatial operator (or linearized spatial operator) is non-coercive, the standard L_2-error bounds based on energy estimates are very pessimistic. In particular, these energy estimates suggest exponential growth in time even for problems which are asymptotically stable and for which the actual error grows at most linearly with time.

In a recent paper [10] space-time adaptive numerical schemes for linear parabolic initial value problems based upon wavelets have been introduced. One key ingredient there is the transformation of the partial differential equation into an equivalent well-conditioned discrete (but still infinite-dimensional) system w.r.t. the wavelet coefficients. In order to show this equivalence, a new proof for the well-posedness of the space-time variational formulation of linear parabolic initial value problems is presented in [10]. This proof contains an explicit lower bound for the inf-sup stability constant. In the context of RBMs, it is well-known that the inverse of the inf-sup-constant multiplied with the (computable) dual norm of the residual form an a-posteriori error estimate in a (Petrov-)Galerkin scheme. A closer investigation and modification of the proof in [10, Theorem 5.1, Appendix A] shows that a space-time inf-sup stability constant — and related appropriate norms — can avoid the "worst–case" energy assumption at each time t (or discrete time level) and instead reflect the coupled temporal behavior over the entire time interval of interest.

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We show in [13] that indeed a space-time formulation can improve reduced basis error bounds: we provide theoretical justification for the symmetric coercive case, and computational evidence for the non-symmetric non-coercive case. We elaborate here on the brief presentation of [13]: we consider in detail the underlying Petrov-Galerkin discretization and associated Crank-Nicolson interpretation; we provide the proofs of the central propositions; we extend the approach and analysis to primal-dual formulations for the output of interest; and finally, we provide numerical convergence results for the inf-sup constant which plays the crucial role in the reduced basis error bounds.

This paper is organized as follows. In Section 2, we investigate the space-time variational problem, in particular its well-posedness also for long time periods. We also show the main difference of our analysis as opposed to more standard techniques using a temporal transformation. Next, we introduce a space-time discretization which leads to a Petrov-Galerkin scheme whose error is analyzed for the particular case of symmetric coercive spatial operators. Section 3 contains the application of our space-time error analysis for the Reduced Basis Method. We give a posteriori error bounds w.r.t. the residual and discuss various issues concerning the numerical realization. We present numerical results in Section 4, in particular for those cases that are not yet covered by our theory, namely convection-diffusion operators as well as asymptotically unstable equations.

2. SPACE-TIME TRUTH SOLUTION

2.1. **Space-Time Formulation.**

Similar to [10], we consider Hilbert spaces $V \hookrightarrow H \hookrightarrow V'$ with inner products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_H$ and induced norms $\|\cdot\|_V$, $\|\cdot\|_H$, a time interval $I := (0,T]$, $T>0$ and $A \in \mathcal{L}(V,V')$ such that $\langle A\phi, \psi\rangle_{V'\times V} = a(\phi,\psi)$ with a bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$. We consider the following problem: Given $g \in L_2(I;V')$, determine u such that

$$\dot{u}(t) + A\,u(t) = g(t) \text{ in } V', \qquad u(0) = 0 \text{ in } H. \tag{2.1}$$

Nonzero initial conditions can easily be treated by slight modifications of the variational form to be introduced next. According to [10] we assume that there exist constants $0 < M_a < \infty$, $\alpha > 0$ and $\lambda \geq 0$ such that for all $\phi, \psi \in V$ we have

$$|a(\psi,\phi)| \leq M_a \,\|\psi\|_V \,\|\phi\|_V, \qquad \text{(boundedness)} \tag{2.2}$$

$$a(\psi,\psi) + \lambda\,\|\psi\|_H^2 \geq \alpha \|\psi\|_V^2. \qquad \text{(Gårding inequality)} \tag{2.3}$$

Note that these assumptions also cover the non-coercive case. For tight a posteriori error estimates for the coercive convection-dominated case, we refer to [15]. In addition, we consider outputs of the form

$$s := \int_I \ell(u(t))\,dt \tag{2.4}$$

for some time-invariant $\ell \in V$. The above setting corresponds to the LTI (linear time invariant) case, but we remark that some of our results can be extended to the LTV (linear time varying) case as well, see for example [17] for Burger's equation as well as [16] for the Boussinesq equations.

In order to formulate the variational form of (2.1), we need some preparation. We use as trial space

$$\mathcal{X} := \{v \in L_2(I;V) : \, v, \dot{v} \in L_2(I;V'), v(0) = 0\} = L_2(I;V) \cap H^1_{(0)}(I;V'),$$

where $H^1_{(0)}(I;V') := \{v \in H^1(I;V') : v(0) = 0\}$ with the (slightly non-standard) norm $\|w\|^2_{\mathcal{X}} := \|w\|^2_{L_2(I;V)} + \|\dot{w}\|^2_{L_2(I;V')} + \|w(T)\|^2_H$ (note: $\mathcal{X} \hookrightarrow C(I;H)$). The test space is $\mathcal{Y} := L_2(I;V)$ with norm $\|v\|_{\mathcal{Y}} := \|v\|_{L_2(I;V)}$.

Remark 2.1. *At a first glance, it seems to be more standard to use the graph norm* $(\|w\|^2_{L_2(I;V)} + \|\dot{w}\|^2_{L_2(I;V')})^{1/2}$ *on* $\mathcal{X}$. *Obviously,* $\|\cdot\|_{\mathcal{X}}$ *defined above is a stronger norm and also allows the control of the solution at the final time* T.

We will use the following abbreviations: $[w,v]_{\mathcal{H}} := \int_I \langle w(t), v(t)\rangle_{V'\times V} dt$ for $w \in L_2(I;V')$, $v \in L_2(I;V)$ (as well as $[w,v]_{\mathcal{H}} := \int_I (w(t),v(t))_H dt$ for $v,w \in L_2(I;H)$) and $\mathcal{A}[w,v] := \int_I a(w(t),v(t))\, dt$ for $v,w \in L_2(I;V)$. Then, defining

$$b(w,v) := [\dot{w},v]_{\mathcal{H}} + \mathcal{A}[w,v], \qquad f(v) := [g,v]_{\mathcal{H}}, \tag{2.5}$$

results in the variational formulation

$$\text{find}\, u \in \mathcal{X} : \qquad b(u,v) = f(v) \qquad \forall v \in \mathcal{Y}. \tag{2.6}$$

The output is again given by (2.4) and can also be formulated as

$$s = J(u) \quad \text{where} \quad J \in \mathcal{X}' \quad \text{reads} \quad J(w) := \int_I \ell(w(t))\, dt,\ w \in \mathcal{X}. \tag{2.7}$$

The well-posedness of (2.6) (under the above assumptions) has been shown in [10, Theorem 5.1, Appendix A]. A more detailed investigation of the proof in [10] shows that the arguments used there can also yield an estimate for the inf-sup constant

$$\beta := \inf_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \frac{b(w,v)}{\|w\|_{\mathcal{X}}\, \|v\|_{\mathcal{Y}}}.$$

We define $\varrho := \sup_{0\neq\phi\in V} \frac{\|\phi\|_H}{\|\phi\|_V}$ and $\beta^*_a := \inf_{\phi\in V} \sup_{\psi\in V} \frac{a(\psi,\phi)}{\|\phi\|_V\, \|\psi\|_V}$; we then have

Proposition 2.2 ([13, Proposition 1])**.** *Assume* (2.2) *and* (2.3)*. Then, we obtain the inf-sup lower bound*

$$\beta \geq \beta^{LB} := \frac{\min\{1, (\alpha - \lambda\varrho^2)\min\{1, M_a^{-2}\}\}}{\max\{1, (\beta^*_a)^{-1}\}\sqrt{2}}.$$

Proof. Let $0 \neq w \in \mathcal{X}$ be given and denote by $A^* : V \to V'$ the adjoint of A. Set $z_w := (A^*)^{-1}\dot{w}$ and $v_w := z_w + w \in \mathcal{Y}$. Then, we have

$$\begin{aligned}\|v_w\|^2_{L_2(I;V)} &\leq 2(\|z_w\|^2_{\mathcal{Y}} + \|w\|^2_{\mathcal{Y}}) \leq 2\big((\beta^*_a)^{-2}\|\dot{w}\|^2_{L_2(I;V')} + \|w\|^2_{\mathcal{Y}})\big) \\ &\leq 2\max\{1, (\beta^*_a)^{-2}\}\|w\|^2_{\mathcal{X}}.\end{aligned} \tag{2.8}$$

In order to bound $b(w,v_w)$ we use the following facts $\|\dot{w}(t)\|_{V'} = \|A^* z_w(t)\|_{V'} \leq M_a\|z_w(t)\|_V$ and thus

$$\begin{aligned}\langle \dot{w}(t), z_w(t)\rangle_{V'\times V} &= a(z_w(t), z_w(t)) \geq \alpha\|\|z_w(t)\|^2_V - \lambda\|z_w(t)\|^2_H \\ &\geq (\alpha - \lambda\varrho^2)\|z_w(t)\|^2_V \geq (\alpha - \lambda\varrho^2)M_a^{-2}\|\dot{w}(t)\|^2_{V'},\end{aligned} \tag{2.9}$$

as well as

$$a(w(t), z_w(t)) = \langle w(t), \dot{w}(t)\rangle_{V\times V'} = \frac{1}{2}\frac{d}{dt}\|w(t)\|^2_H \tag{2.10}$$

to obtain (recalling that $w(0)=0$)

$$\begin{aligned} b(w,v_w) &= \int_I \langle \dot{w}(t), z_w(t)\rangle_{V'\times V}\, dt + \int_I \langle \dot{w}(t), w(t)\rangle_{V'\times V} dt \\ &\quad + \int_I a(w(t), z_w(t))\, dt + \int_I a(w(t), w(t))\, dt \\ &\geq (\alpha - \lambda\varrho^2) M_a^{-2} \|\dot{w}\|^2_{L_2(I;V')} + \frac{1}{2}\int_0^T \frac{d}{dt}\|w(t)\|_H^2\, dt \\ &\quad + \frac{1}{2}\int_0^T \frac{d}{dt}\|w(t)\|_H^2\, dt + (\alpha - \lambda\varrho^2)\|w(t)\|^2_{L_2(I;V)} \\ &\geq (\alpha - \lambda\varrho^2)\min\{1, M_a^{-2}\}(\|\dot{w}\|^2_{L_2(I;V')} + \|w\|^2_{L_2(I;V)}) + \|w(T)\|_H^2 \\ &\geq \min\{(\alpha - \lambda\varrho^2)\,\min\{1, M_a^{-2}\}, 1\}\|w\|^2_{\mathcal{X}} \geq \beta^{\rm LB}\|w\|_{\mathcal{X}}\|v_w\|_{\mathcal{Y}}, \end{aligned}$$

where the last step follows from (2.8). □

Remark 2.3. *Note that β^{LB} does not depend on the final time. However, the estimate is only meaningful if $\alpha \geq \lambda\varrho^2$, i.e., if the system is coercive. In the non-coercive case,* (2.1) *is often transformed as described in Section 2.3 below.*

Remark 2.4. *If we would use the graph norm for $\mathcal{X}$, the above proof yields an inf-sup lower bound of $\frac{(\alpha-\lambda\varrho^2)\min\{1,M_a^{-2}\}}{\max\{1,(\beta_a^*)^{-1}\}\sqrt{2}}$.*

2.2. **The heat equation.** The heat equation is a special case of (2.1), where

$$A = -\Delta, \quad V = H_0^1(\Omega), \quad H = L_2(\Omega), \quad \|\phi\|_V^2 = a(\phi,\phi) = \|\nabla\phi\|^2_{L_2(\Omega)}.$$

Thus, we have $M_a = 1$, $\lambda = 0$, $\alpha = 1$ and $\beta_a^* = 1$. Thus, Proposition 2.2 would result in a lower bound of $\frac{1}{\sqrt{2}}$. A slight modification of the proof, however, allows to improve this lower bound.

Corollary 2.5. *For the heat equation, it holds $\beta \geq 1$.*

Proof. Given $0 \neq w \in \mathcal{X}$, we choose as above $v_w := z_w + w \in \mathcal{Y}$ with $z_w := A^{-1}\dot{w}$. Then,

$$\|v_w\|^2_{L_2(I;V)} = \|z_w\|^2_{L_2(I;V)} + \|w\|^2_{L_2(I;V)} + 2\int_I (z_w(t), w(t))_V\, dt.$$

Since $\|z_w\|_{L_2(I;V)} = \|A^{-1}\dot{w}\|_{L_2(I;V)} = \|\dot{w}\|_{L_2(I;V')}$ and recalling that $a(z_w(t), v(t)) = \langle \dot{w}(t), v(t)\rangle_{V'\times V}$ for all $v(t) \in V$, we obtain

$$(z_w(t), w(t))_V = a(z_w(t), w(t)) = \langle \dot{w}(t), w(t)\rangle_{V'\times V} = \frac{1}{2}\frac{d}{dt}\|w(t)\|_H^2,$$

so that

$$\begin{aligned} \|v_w\|^2_{L_2(I;V)} &= \|A^{-1}\dot{w} + w\|^2_{L_2(I;V)} \\ &= \|\dot{w}\|^2_{L_2(I;V')} + \|w\|^2_{L_2(I;V)} + \|w(T)\|_H^2 = \|w\|^2_{\mathcal{X}}. \end{aligned} \tag{2.11}$$

The rest of the proof remains the same so that we arrive at $b(w,v_w) \geq \|w\|^2_{\mathcal{X}} = \|w\|_{\mathcal{X}}\,\|v_w\|_{\mathcal{Y}}$. □

We can go even a step further.

Proposition 2.6. *For the heat equation, it holds $\beta = \gamma = 1$, where γ is the continuity constant defined as $\gamma := \sup_{w\in\mathcal{X}}\sup_{v\in\mathcal{Y}} \frac{b(w,v)}{\|w\|_{\mathcal{X}}\,\|v\|_{\mathcal{Y}}}$.*

Proof. For $w \in \mathcal{X}$ and $v \in \mathcal{Y}$ we have $b(w, v) = \int_I a(A^{-1}\dot{w}(t) + w(t), v(t))\, dt$. Given $v \in \mathcal{Y}$, we have $Av \in L_2(I; V') = \mathcal{Y}'$ and Corollary 2.5 ensures that there exists a unique $z \in \mathcal{X}$ such that $\dot{z} + Az = Av$, i.e, $v = A^{-1}\dot{z} + z$. Then, we have

$$\begin{aligned}\sup_{v\in\mathcal{Y}} \frac{b(w,v)}{\|v\|_{\mathcal{Y}}} &= \sup_{z\in\mathcal{X}} \frac{b(w, A^{-1}\dot{z} + z)}{\|A^{-1}\dot{z} + z\|_{\mathcal{Y}}} \\ &= \sup_{z\in\mathcal{X}} \frac{\int_I a(A^{-1}\dot{w}(t) + w(t), A^{-1}\dot{z}(t) + z(t))\, dt}{\|A^{-1}\dot{z} + z\|_{\mathcal{Y}}} \\ &= \|A^{-1}\dot{w} + w\|_{\mathcal{Y}} = \|w\|_{\mathcal{X}},\end{aligned}$$

where the last step is shown in (2.11). The claim is thus proven. □

2.3. **Using temporal transformation.** Another possibility to derive a lower inf-sup-bound is the transformation of the initial value problem (2.1) in the following (standard and well-known) way. In view of the Gårding inequality (2.3), setting $\hat{u}(t) := e^{-\lambda t}u(t)$, $\hat{v}(t) := e^{\lambda t}v(t)$ and $\hat{g}(t) := e^{-\lambda t}g(t)$ solves the variational problem

$$\hat{b}(\hat{w}, \hat{v}) = \hat{f}(\hat{v}), \qquad \forall \hat{v} \in \mathcal{Y},$$

where

$$\hat{b}(\hat{w}, \hat{v}) := \int_0^T \big\langle \frac{d}{dt}\hat{w}(t), \hat{v}(t)\big\rangle_{V'\times V}\, dt + \int_0^T \hat{a}(\hat{w}(t), \hat{v}(t))\, dt$$

as well as $\hat{a}(\hat{w}(t), \hat{v}(t)) := a(\hat{w}(t), \hat{v}(t)) + \lambda(\hat{w}(t), \hat{v}(t))_H$ and for the right-hand side $\hat{f}(\hat{v}) := \int_I \langle \hat{g}(t), \hat{v}(t)\rangle_{V'\times V}\, dt$. Note, that the form $\hat{a}$ fulfills (2.3) with $\lambda = 0$ which gives rise to the following lower inf-sup-bound

Corollary 2.7. *Under the above assumptions, we get the following lower bound for the inf-sup-constant*

$$\beta \geq \hat{\beta}_{LB} := \frac{e^{-2\lambda T}}{\max\{\sqrt{1 + 2\lambda^2\varrho^4}, \sqrt{2}\}} \times \frac{\min\{1, \alpha\min\{1, M_a^{-2}\}\}}{\max\{1, (\beta_a^*)^{-1}\}\sqrt{2}}. \tag{2.12}$$

Proof. It is readily seen that $\hat{b}(\hat{w}, \hat{v}) = b(w, v)$ with the above transformations, so that it remains to estimate the norms. It is known from [10, Appendix A] that

$$\|w\|_{\mathcal{X}} \leq e^{\lambda T}\, \max\{\sqrt{1 + 2\lambda^2\varrho^4}, \sqrt{2}\}\|\hat{w}\|_{\mathcal{X}}, \quad \|v\|_{\mathcal{Y}} \leq e^{\lambda T}\|\hat{v}\|_{\mathcal{Y}}.$$

This implies that

$$\begin{aligned}\inf_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \frac{b(w,v)}{\|w\|_{\mathcal{X}}\, \|v\|_{\mathcal{Y}}} &= \inf_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \frac{\hat{b}(\hat{w},\hat{v})}{\|w\|_{\mathcal{X}}\, \|v\|_{\mathcal{Y}}} \\ &\geq e^{-2\lambda T}\, \max\{\sqrt{1 + 2\lambda^2\rho^4}, \sqrt{2}\}^{-1} \inf_{w\in\mathcal{X}} \sup_{v\in\mathcal{Y}} \frac{\hat{b}(\hat{w},\hat{v})}{\|\hat{w}\|_{\mathcal{X}}\, \|\hat{v}\|_{\mathcal{Y}}}.\end{aligned}$$

The result then follows from Proposition 2.2. □

Remark 2.8. *Obviously this approach yields an inf-sup bound that behaves as* $e^{-\lambda T}$ *— often extremely pessimistic and clearly unsuitable for error estimation in long-time integration.*

2.4. **Petrov-Galerkin Truth Approximation.** Let $\mathcal{X}_\delta \subset \mathcal{X}$, $\mathcal{Y}_\delta \subset \mathcal{Y}$ be finite dimensional subspaces and $u_\delta \in \mathcal{X}_\delta$ the discrete approximation of (2.6), i.e.,

$$b(u_\delta, v_\delta) = f(v_\delta), \qquad \forall v_\delta \in \mathcal{Y}_\delta, \tag{2.13}$$

$s_\delta = \int_0^T \ell(u_\delta(t))\, dt$. Henceforth, we concentrate on the case $H = L_2(\Omega)$, $V = H_0^1(\Omega)$. Let $\mathcal{X}_\delta = S_{\Delta t} \otimes V_h$, $\mathcal{Y}_\delta = Q_{\Delta t} \otimes V_h$, $\delta = (\Delta t, h)$, where $S_{\Delta t}$, V_h are piecewise linear and $Q_{\Delta t}$ piecewise constant finite elements with respect to triangulations $\mathcal{T}_h^{\text{space}}$ in space and $\mathcal{T}_{\Delta t}^{\text{time}} \equiv \{t^{k-1} \equiv (k-1)\Delta t < t \leq k\,\Delta t \equiv t^k, 1 \leq k \leq K\}$ in time for $\Delta t := T/K$.

Let $S_{\Delta t} = \text{span}\{\sigma^1, \ldots, \sigma^K\}$, where σ^k is the (interpolatory) hat-function with the nodes t^{k-1}, t^k and t^{k+1} (resp. truncated for $k = K$) and $Q_{\Delta t} = \text{span}\{\tau^1, \ldots, \tau^K\}$, where $\tau^k = \chi_{I^k}$, the characteristic function on $I^k := (t^{k-1}, t^k)$. Finally, let $V_h = \text{span}\{\phi_1, \ldots, \phi_{n_h}\}$ e.g. be the nodal basis w.r.t. $\mathcal{T}_h^{\text{space}}$. For any given $w_\delta = \sum_{k=1}^K \sum_{i=1}^{n_h} w_i^k \sigma^k \otimes \phi_i \in \mathcal{X}_\delta$ and $v_\delta = \sum_{\ell=1}^K \sum_{j=1}^{n_h} v_j^\ell\, \tau^k \otimes \phi_j$ (with coefficients w_i^k and v_j^ℓ) we obtain

$$\begin{aligned} b(w_\delta, v_\delta) &= \int_I \big(\langle \dot{w}_\delta(t), v_\delta(t)\rangle_{V' \times V} + a(w_\delta(t), v_\delta(t))\big)\, dt \\ &= \sum_{k,\ell=1}^{K} \sum_{i,j=1}^{n_h} w_k^i v_\ell^j \big[(\dot\sigma^k, \tau^\ell)_{L_2(I)}\, (\phi_i, \phi_j)_H + (\sigma^k, \tau^\ell)_{L_2(I)}\, a(\phi_i, \phi_j)\big] \\ &= \mathbf{w}_\delta^T \mathbf{B}_\delta \mathbf{v}_\delta, \end{aligned}$$

where

$$\mathbf{B}_\delta := \mathbf{N}_{\Delta t}^{\text{time}} \otimes \mathbf{M}_h^{\text{space}} + \mathbf{M}_{\Delta t}^{\text{time}} \otimes \mathbf{A}_h^{\text{space}} \tag{2.14}$$

and $\mathbf{M}_h^{\text{space}} := [(\phi_i, \phi_j)_{L_2(\Omega)}]_{i,j=1,\ldots,n_h}$, $\mathbf{M}_{\Delta t}^{\text{time}} := [(\sigma^k, \tau^\ell)_{L_2(I)}]_{k,\ell=1,\ldots,K}$ are the spatial and temporal mass matrices as well as $\mathbf{N}_{\Delta t}^{\text{time}} := [(\dot\sigma^k, \tau^\ell)_{L_2(I)}]_{k,\ell=1,\ldots,K}$ and $\mathbf{A}_h^{\text{space}} := [a(\phi_i, \phi_j)]_{i,j=1,\ldots,n_h}$. For our particular spaces we obtain (denoting by $\delta_{k,\ell}$ the discrete Kronecker delta)

$$\begin{aligned} (\dot\sigma^k, \tau^\ell)_{L_2(I)} &= \delta_{k,\ell} - \delta_{k+1,\ell}, \qquad (\sigma^k, \tau^\ell)_{L_2(I)} = \frac{\Delta t}{2}(\delta_{k,\ell} + \delta_{k+1,\ell}), \\ b(w_\delta, \tau^\ell \otimes \phi_j) &= \sum_{i=1}^{n_h} \Big[(w_i^\ell - w_i^{\ell-1})(\phi_i, \phi_j)_H + \frac{\Delta t}{2}(w_i^\ell + w_i^{\ell-1})\, a(\phi_i, \phi_j)\Big] \\ &= \Delta t \Big[\mathbf{M}_h^{\text{space}} \frac{1}{\Delta t}(\mathbf{w}^\ell - \mathbf{w}^{\ell-1}) + \mathbf{A}_h^{\text{space}} \mathbf{w}^{\ell-1/2}\Big], \end{aligned}$$

where $\mathbf{w}^\ell := (w_i^\ell)_{i=1,\ldots,n_h}$, $w_i^{\ell-1/2} := \frac{1}{2}(w_i^\ell + w_i^{\ell-1})$ and $\mathbf{w}^{\ell-1/2}$ accordingly. If we use a trapezoidal approximation of the right-hand side temporal integration

$$\begin{aligned} f(\tau^\ell \otimes \phi_j) &= \int_0^T \langle g(t), \tau^\ell \otimes \phi_j\rangle_{V' \times V}\, dt \\ &\approx \frac{\Delta t}{2}\langle g(t^{\ell-1}) + g(t^\ell), \phi_j\rangle_{V' \times V} = \frac{\Delta t}{2}(\mathbf{g}^{\ell-1} + \mathbf{g}^\ell)_j = \Delta t\ \mathbf{g}_j^{\ell-1/2}, \end{aligned}$$

where $\mathbf{g}^\ell = (\langle g(t^\ell), \phi_j\rangle_{V' \times V})_{j=1,\ldots,n_h}$, then we can rewrite (2.13) as

$$\frac{1}{\Delta t}\mathbf{M}_h^{\text{space}}(\mathbf{w}^\ell - \mathbf{w}^{\ell-1}) + \mathbf{A}_h^{\text{space}} \mathbf{w}^{\ell-1/2} = \mathbf{g}^{\ell-1/2}, \qquad \mathbf{w}^0 := 0, \tag{2.15}$$

which is nothing else then the well-known Crank–Nicolson (CN) scheme; hence, we can derive error bounds for the CN scheme via our space-time formulation.

For the analysis we introduce a different norm on $\mathcal{X}$ associated with our temporal discretization: For $w \in \mathcal{X}$ set $\bar{w}^k := (\Delta t)^{-1} \int_{I^k} w(t)\, dt \in V$ and $\bar{w} := \sum_{k=1}^K \chi_{I^k} \otimes \bar{w}^i \in L_2(I;V)$; then, set

$$|||w|||_{\mathcal{X},\delta}^2 := \|\dot{w}\|_{L_2(I;V')}^2 + \|\bar{w}\|_{L_2(I;V)}^2 + \|w(T)\|_H^2$$

and the inf-sup parameter as well as the stability parameter

$$\beta_\delta := \inf_{w_\delta \in \mathcal{X}_\delta} \sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(w_\delta, v_\delta)}{|||w_\delta|||_{\mathcal{X},\delta}\, \|v_\delta\|_{\mathcal{Y}}}, \qquad \gamma_\delta := \sup_{w_\delta \in \mathcal{X}_\delta} \sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(w_\delta, v_\delta)}{|||w_\delta|||_{\mathcal{X},\delta}\, \|v_\delta\|_{\mathcal{Y}}}.$$

Note this local-average-in-time norm can be motivated by the corresponding "natural" norm associated with the Crank-Nicolson discretization: Upon multiplication of (2.15) by $\mathbf{w}^{\ell-1/2}$ we obtain $\mathbf{w}^{\ell-1/2}\mathbf{A}_h^{\text{space}}\mathbf{w}^{\ell-1/2}$ for the $L_2(I;V)$-contribution to the energy. In the space-time context the corresponding result is provided in [13].

Proposition 2.9 ([13, Proposition 3])**.** *Let $a(\cdot,\cdot)$ be symmetric, bounded and coercive and set $\|\phi\|_V^2 := a(\phi,\phi)$, $\phi \in V$; then we have $\beta_\delta = \gamma_\delta = 1$.*

Proof. Since $v_\delta \in \mathcal{Y}_\delta$ is piecewise constant in time, we have $\int_I a(w(t), v_\delta(t))\, dt = \int_I a(\bar{w}(t), v_\delta(t))\, dt$ for all $w \in \mathcal{X}$. Hence, $b(w_\delta, v_\delta) = \int_I a(A_h^{-1}\dot{w}_\delta(t) + \bar{w}_\delta(t), v_\delta(t))\, dt$, where $z_\delta(t) := A_h^{-1}\dot{w}_\delta(t)$ is defined by $a(z_\delta(t), \phi_h) = \langle \dot{w}_\delta(t), \phi_h\rangle_{V'\times V}$ for all $\phi_h \in V_h$. Note that for $\tilde{v} \in V'$ we have $\|A_h^{-1}\tilde{v}\|_V^2 = a(A_h^{-1}\tilde{v}, A_h^{-1}\tilde{v}) = \langle \tilde{v}, A_h^{-1}\tilde{v}\rangle_{V'\times V} = \|\tilde{v}\|_{V'}^2$. We will prove later that for all $v_\delta \in \mathcal{Y}_\delta$ there exists a unique $z_\delta \in \mathcal{X}_\delta$ such that

$$\int_I a(A_h^{-1}\dot{z}_\delta(t) + \bar{z}_\delta(t), q_\delta(t))\, dt = \int_I a(v_\delta(t), q_\delta(t))\, dt \qquad \forall q_\delta \in \mathcal{Y}_\delta. \tag{2.16}$$

Note that $v_\delta := A_h^{-1}\dot{z}_\delta + \bar{z}_\delta \in \mathcal{Y}_\delta$ for $z_\delta \in \mathcal{X}_\delta$. Hence,

$$\begin{aligned}
\sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(w_\delta, v_\delta)}{\|v_\delta\|_{\mathcal{Y}}} &= \sup_{z_\delta \in \mathcal{X}_\delta} \frac{b(w_\delta, A_h^{-1}\dot{z}_\delta + \bar{z}_\delta)}{\|A_h^{-1}\dot{z}_\delta + \bar{z}_\delta\|_{\mathcal{Y}}} \\
&= \sup_{z_\delta \in \mathcal{X}_\delta} \frac{\int_I a(A_h^{-1}\dot{w}_\delta + \bar{w}_\delta, A_h^{-1}\dot{z}_\delta + \bar{z}_\delta)\, dt}{\|A_h^{-1}\dot{z}_\delta + \bar{z}_\delta\|_{\mathcal{Y}}} = \|A_h^{-1}\dot{w}_\delta + \bar{w}_\delta\|_{\mathcal{Y}}
\end{aligned}$$

by the Cauchy-Schwarz inequality and choosing $z_\delta = w_\delta$. Next,

$$\begin{aligned}
\|A_h^{-1}\dot{w}_\delta + \bar{w}_\delta\|_{\mathcal{Y}}^2 &= \|A_h^{-1}\dot{w}_\delta\|_{\mathcal{Y}}^2 + \|\bar{w}_\delta\|_{\mathcal{Y}}^2 + 2\int_I \langle \dot{w}_\delta(t), \bar{w}_\delta(t)\rangle_{V'\times V}\, dt \\
&= \|\dot{w}_\delta\|_{L_2(I;V')}^2 + \|\bar{w}_\delta\|_{L_2(I;V)}^2 + \|w_\delta(T)\|_H^2 = |||w_\delta|||_{\mathcal{X},\delta}^2,
\end{aligned}$$

so that $\sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(w_\delta, v_\delta)}{\|v_\delta\|_{\mathcal{Y}}} = |||w_\delta|||_{\mathcal{X},\delta}$ which implies $\beta_\delta = \gamma_\delta = 1$.

It remains to prove (2.16). Let $\eta_j > 0$, $e_j \in \mathbb{R}^{n_h}$, $j = 1, \dots, n_h$, be the eigenvalues and normalized eigenvectors of A_h, i.e.,

$$a(e_j, \phi_h) = \eta_j\, (e_j, \phi_h)_H \qquad \forall \phi_h \in V_h, \qquad \|e_j\|_H = 1,\ 1 \le j \le n_h.$$

Given $\mathcal{Y}_\delta \ni v_\delta = \sum_{k=1}^K v^k\, \tau^k$, $v^k = \sum_{j=1}^{n_h} v_j^k\, e_j \in V_h$, determine ζ_j^k, $k = 1, \dots, K$, $j = 1, \dots, n_h$ as the unique solution of the difference equation

$$\zeta_j^0 = 0, \qquad \frac{1}{\Delta t}(\zeta_j^k - \zeta_j^{k-1}) + \frac{\lambda_j}{2}(\zeta_j^k + \zeta_j^{k-1}) = \eta_j\, v_j^k,\ k = 1, \dots, K. \tag{2.17}$$

Then, define $z_\delta := \sum_{k=1}^K \sum_{j=1}^{n_h} \zeta_j^k \, e_j \, \sigma^k \in \mathcal{X}_\delta$, so that

$$\bar{z}_\delta = \sum_{k=1}^K \bar{z}_\delta^k \, \chi_{I^k} = \sum_{k=1}^K \frac{\Delta t}{2}(z^k + z^{k-1})\tau^k, \quad z^k := z_\delta(t^k),$$

since z_δ is piecewise linear in time. Then we obtain for any $q_\delta \in \mathcal{Y}_\delta$, $q_\delta = \sum_{k=1}^K q^k \tau^k$, $q^k = q_\delta(t^k)$

$$\begin{aligned}
\int_I a(v_\delta(t), q_\delta(t))\, dt &= \sum_{k,\ell=1}^K a(v^k, q^\ell) \int_I \tau^k(t)\, \tau^\ell(t)\, dt = \sum_{k=1}^K \Delta t \; a(v^k, q^k) \\
&= \sum_{k=1}^K \sum_{j=1}^{n_h} \Delta t \, v_j^k \, \eta_j \, (e_j, q^k)_H \\
&= \sum_{k=1}^K \sum_{j=1}^{n_h} \Delta t \, (e_j, q^k)_H \Big[\frac{1}{\Delta t}(\zeta_j^k - \zeta_j^{k-1}) + \eta_j \, \zeta_j^{k-1/2} \Big] \\
&= \sum_{k=1}^K \Big(\sum_{j=1}^{n_h} (\zeta_j^k - \zeta_j^{k-1}) e_j, q^k \Big)_H + \Delta t \sum_{k=1}^K \sum_{j=1}^{n_h} a(\zeta_j^{k-1/2} e_j, q^k) \\
&= \int_I \langle \dot{z}_\delta(t), q_\delta(t) \rangle_{V' \times V}\, dt + \int_I a(\bar{z}_\delta(t), q_\delta(t))\, dt.
\end{aligned}$$

This proves the existence in (2.16). The uniqueness is seen as follows. Let $z_\delta, w_\delta \in \mathcal{X}_\delta$ be two solutions of (2.16), then

$$\int_I a(A_h^{-1}(\dot{z}_\delta(t) - \dot{w}_\delta(t)) + \bar{z}_\delta(t) - \bar{w}_\delta(t), q_\delta(t))\, dt = 0 \quad \forall q_\delta \in \mathcal{Y}_\delta.$$

By using the first argument as test function we arrive at $\|\dot{z}_\delta - \dot{w}_\delta\|^2_{L_2(I;V')} + \|\bar{z}_\delta - \bar{w}_\delta\|^2_{L_2(I;V)} + \|z_\delta(T) - w_\delta(T)\|_H^2 = 0$, which shows the uniqueness in $\mathcal{X}_\delta$. ☐

Remark 2.10. *We may rephrase Proposition 2.9 also in the following way:*

$$\sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(w_\delta, v_\delta)}{\|v_\delta\|_{\mathcal{Y}}} = |||w_\delta|||_{\mathcal{X},\delta}, \quad w_\delta \in \mathcal{X}_\delta. \tag{2.18}$$

Moreover, the proof also shows that

$$\forall\, 0 \neq w_\delta \in \mathcal{X}_\delta \quad \exists v_\delta \in \mathcal{Y}_\delta : \qquad \frac{b(w_\delta, v_\delta)}{\|v_\delta\|_{\mathcal{Y}}} = |||w_\delta|||_{\mathcal{X},\delta} \neq 0. \tag{2.19}$$

Remark 2.11. *Proposition 2.9 also shows the well-posedness of the discrete problem with continuity and inf-sup constant being unity.*

For later purpose, we consider also the dual inf-sup parameter defined as

$$\beta_\delta^* := \inf_{v_\delta \in \mathcal{Y}_\delta} \sup_{w_\delta \in \mathcal{X}_\delta} \frac{b(w_\delta, v_\delta)}{|||w_\delta|||_{\mathcal{X},\delta} \, \|v_\delta\|_{\mathcal{Y}}}.$$

Proposition 2.12. *Under the hypotheses of Proposition 2.9, we have $\beta_\delta^* = \beta_\delta = 1$.*

Proof. We use Nečas' theorem [6, Theorem 3.3] that shows that (2.18) and (2.19) are equivalent to $\beta_\delta^* = \beta_\delta = 1$. ☐

3. The Reduced Basis Method (RBM)

3.1. **Parameter-dependence.** Now, let $\mu \in \mathcal{D} \subseteq \mathbb{R}^P$ be a parameter vector and $A = A(\mu)$ a parameter-dependent linear partial differential operator. It is fairly standard to assume that $A(\mu)$ is induced by a bilinear form $a(\cdot,\cdot;\mu)$ that is affine w.r.t. the parameter, i.e., there exist functions $\theta_q^a(\mu)$ and bilinear forms $a_q(\cdot,\cdot)$ such that

$$a(\psi,\phi;\mu) = \sum_{q=1}^{Q^a} \theta_q^a(\mu)\, a_q(\psi,\phi), \qquad \mu \in \mathcal{D},\ \psi,\phi \in V. \tag{3.1}$$

We obtain the parameter-dependent space-time bilinear form

$$b(w,v;\mu) = [\dot{w},v;\mu]_{\mathcal{H}} + \mathcal{A}[w,v;\mu], \quad \text{with } \mathcal{A}[w,v;\mu] = \int_I a(w(t),v(t);\mu)\, dt,$$

where $[\cdot,\cdot;\mu]_{\mathcal{H}}$ is a parameter-dependent version of $[\cdot,\cdot]_{\mathcal{H}}$ with a similar expansion as in (3.1), such that we derive an affine decomposition according to

$$b(w,v;\mu) = \sum_{q=1}^{Q} \theta_q(\mu)\, b_q(w,v).$$

Also the right-hand side may depend on the parameter and is also assumed to be affine in functions of the parameter, i.e.,

$$f(v;\mu) = \sum_{q=1}^{Q} \theta_q^f(\mu)\, f_q(v), \qquad \mu \in \mathcal{D},\ v \in \mathcal{Y}. \tag{3.2}$$

If (3.1,3.2) are not satisfied, it is fairly standard to construct an approximation via the *Empirical Interpolation Method* (EIM), [1, 12].

The parameter-dependent version of (2.6) then reads

$$u(\mu) \in \mathcal{X}: \qquad b(u(\mu),v;\mu) = f(v;\mu) \ \ \forall v \in \mathcal{Y}. \tag{3.3}$$

The output reads $s(\mu) := \int_I \ell(u(t;\mu))\, dt$. The truth approximations are then fairly standard, i.e.,

$$u_\delta(\mu) \in \mathcal{X}_\delta: \qquad b(u_\delta(\mu),v_\delta;\mu) = f(v_\delta;\mu) \ \ \forall v_\delta \in \mathcal{Y}_\delta, \tag{3.4}$$

and the output reads $s_\delta(\mu) := \int_I \ell(u_\delta(t;\mu))\, dt = J(u_\delta(\mu))$. Defining

$$\gamma_\delta(\mu) := \sup_{w_\delta \in \mathcal{X}_\delta} \sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(w_\delta,v_\delta;\mu)}{|||w_\delta|||_{\mathcal{X},\delta}\, \|v_\delta\|_{\mathcal{Y}}}, \quad \beta_\delta(\mu) := \inf_{w_\delta \in \mathcal{X}_\delta} \sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(w_\delta,v_\delta;\mu)}{|||w_\delta|||_{\mathcal{X},\delta}\, \|v_\delta\|_{\mathcal{Y}}} \tag{3.5}$$

it is well-known (see also [10]) from the Babuška-Aziz theorem that (3.3) is well-posed for all $\mu \in \mathcal{D}$ provided that the following three properties hold

$$\text{(i) } \gamma_\delta(\mu) \le \gamma_\delta^{\mathrm{UB}} < \infty, \qquad \text{(ii) } \beta_\delta(\mu) \ge \beta_\delta^{\mathrm{LB}} > 0, \qquad \text{(iii) } b(\cdot,\cdot;\mu) \text{ is surjective.}$$

3.2. **RB error bounds.** We introduce a standard Reduced Basis (RB) approximation [3, 8, 9] for the Crank–Nicolson interpretation (2.15) of our discrete problem. Let $V_N := \operatorname{span}\{\xi_1,\ldots,\xi_N\} \subset V_h$ be an RB space provided for example by the POD-Greedy procedure of [4]. Then, set $\mathcal{X}_{\Delta t,N} := S_{\Delta t} \otimes V_N$, $\mathcal{Y}_{\Delta t,N} := Q_{\Delta t} \otimes V_N$ and let $u_N(\mu) \in \mathcal{X}_{\Delta t,N}$ denote the unique solution of

$$b(u_N(\mu),v_N;\mu) = f(v_N;\mu) \qquad \forall v_N \in \mathcal{Y}_{\Delta t,N}. \tag{3.6}$$

The RB output is then given by

$$s_N(\mu) := J(u_N(\mu)) = \int_I \ell(u_N(t;\mu))dt \left(= \int_I \ell(\bar{u}_N(t;\mu))dt \right).$$

(It is possible, alternatively, to consider a space–time RB approximation as well [11].)

We define the common RB-quantities, namely the *error* $e_N(\mu) := u_\delta(\mu) - u_N(\mu)$, the *residual*

$$r_N(v;\mu) := f(v;\mu) - b(u_N(\mu), v;\mu) = b(e_N(\mu), v;\mu), \quad v \in \mathcal{Y}_\delta,$$

the *Riesz representation* $\hat{r}_N(\mu) \in \mathcal{Y}_\delta$ (not in $\mathcal{X}_\delta$!) as

$$(\hat{r}_N(\mu), v)_{\mathcal{Y}} = r_N(v;\mu), \quad v \in \mathcal{Y}_\delta$$

and $\|\hat{r}_N(\mu)\|_{\mathcal{Y}} = \|r_N(\mu)\|_{\mathcal{Y}'}$. The 'truth dual norm' on $\mathcal{X}_\delta'$ is defined as

$$|||\tilde{J}|||_{\mathcal{X}',\delta} := \sup_{w \in \mathcal{X}_\delta} \frac{\tilde{J}(w)}{|||w|||_{\mathcal{X},\delta}}, \qquad \tilde{J} \in \mathcal{X}_\delta'.$$

It is then simple [9] to demonstrate

Proposition 3.1. *The following estimates hold*

(a) $|||u_\delta(\mu) - u_N(\mu)|||_{\mathcal{X},\delta} \leq \frac{\|r_N(\mu)\|_{\mathcal{Y}'}}{\beta_\delta^{\mathrm{LB}}}$;

(b) $|s_\delta(\mu) - s_N(\mu)| \leq \frac{\sqrt{T}}{\beta_\delta^{LB}} \|\ell\|_{V'} \|r_N(\mu)\|_{\mathcal{Y}'}$.

Proof. The proof follows standard arguments

$$\beta_\delta^{\mathrm{LB}} |||u_\delta(\mu) - u_N(\mu)|||_{\mathcal{X},\delta} \leq \sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(e_N(\mu), v_\delta(\mu))}{\|v_\delta\|_{\mathcal{Y}}} = \sup_{v_\delta \in \mathcal{Y}_\delta} \frac{r_N(v_\delta;\mu)}{\|v_\delta\|_{\mathcal{Y}}} = \|r_N(\mu)\|_{\mathcal{Y}'}$$

as well as (noting that $\int_I \ell(u_\delta(t;\mu))dt = \int_I \ell(\overline{u}_\delta(t;\mu))dt$ for our time-invariant output and choice of discrete space)

$$\begin{aligned} |s_\delta(\mu) - s_N(\mu)| &= \left| \int_I \ell(\bar{u}_\delta(t;\mu)) - \ell(\bar{u}_N(t;\mu))\, dt \right| \\ &\leq \int_I \|\ell\|_{V'} \, \|\bar{u}_\delta(t;\mu) - \bar{u}_N(t;\mu)\|_V \, dt \\ &\leq \|\ell\|_{V'} \sqrt{T} \|\bar{u}_\delta(\mu) - \bar{u}_N(\mu)\|_{L_2(I;V)} \leq \sqrt{T} \|\ell\|_{V'} \, |||e_N(\mu)|||_{\mathcal{X},\delta} \end{aligned}$$

which –combined with (a)– proves (b). ☐

The utility of these *a posteriori* error bounds is critically dependent on the dependence of β_δ as a function of the parameter μ and final time T, $\beta_\delta(\mu;T)$. We will investigate this dependence in our numerical experiments described in Section 4 below.

Remark 3.2. *We have proven that the error estimate is* exact *for the case of the heat equation, which means that the effectivity is optimal. In the parameter-dependent case, there are two issues: (1) one needs a lower bound for the inf-sup-constant and (2) the energy norm (which is μ-dependent) cannot be used in online computations, and hence one also needs a lower bound for a coercivity constant. Thus the error bound will deviate from optimal.*

For the case of a non–symmetric or a non–coercive operator — the latter case is of greatest interest in the current context — we do not yet have any theoretical *results for the effectivity. However, in practice* [17] *the error bounds are reasonably sharp.*

Primal-dual formulation. The estimate (b) in Proposition 3.1 is not completely satisfying since the error estimator grows with respect to time. In order to overcome this issue, we consider a dual problem. The original, truth and RB dual problem, respectively, read

$$\text{find } z(\mu) \in \mathcal{Y}: \qquad b(w, z(\mu); \mu) = -J(w) \qquad \forall\, w \in \mathcal{X}, \tag{3.7}$$

$$\text{find } z_\delta(\mu) \in \mathcal{Y}_\delta: \qquad b(w_\delta, z_\delta(\mu); \mu) = -J(w_\delta) \qquad \forall\, w_\delta \in \mathcal{X}_\delta, \tag{3.8}$$

$$\text{find } z_{\tilde{N}}(\mu) \in \tilde{\mathcal{Y}}_{\Delta t, \tilde{N}}: \; b(w_{\tilde{N}}, z_{\tilde{N}}(\mu); \mu) = -J(w_{\tilde{N}}) \quad \forall\, w_{\tilde{N}} \in \tilde{\mathcal{X}}_{\Delta t, \tilde{N}}, \tag{3.9}$$

where $\tilde{\mathcal{X}}_{\Delta t, \tilde{N}} := S_{\Delta t} \otimes \tilde{V}_{\tilde{N}}$, $\tilde{\mathcal{Y}}_{\Delta t, \tilde{N}} := Q_{\Delta t} \otimes \tilde{V}_{\tilde{N}}$ and $\tilde{V}_{\tilde{N}} \subset V_h$ is a spatial RB-space also possibly different from V_N. The dual RB residual is defined as $\tilde{r}_{\tilde{N}}(w; \mu) := -J(w) - b(w, z_{\tilde{N}}(\mu); \mu)$ for $w \in \mathcal{X}$, i.e., $\tilde{r}_{\tilde{N}}(\mu) := \tilde{r}_{\tilde{N}}(\cdot; \mu) \in \mathcal{X}'$ and the dual error as $\tilde{e}_{\tilde{N}}(\mu) := z_\delta(\mu) - z_{\tilde{N}}(\mu)$. Finally, we define the RB output in this primal-dual setting as

$$s_N(\mu) := J(u_N(\mu)) - r_N(z_{\tilde{N}}(\mu)).$$

Then, standard RB-arguments yield:

Proposition 3.3. *The following estimates hold*

(a) $\|z_\delta(\mu) - z_{\tilde{N}}(\mu)\|_{\mathcal{Y}} \leq \frac{1}{\beta_{LB}} |||\tilde{r}_{\tilde{N}}(\mu)|||_{\mathcal{X}', \delta}$;

(b) $|s_\delta(\mu) - s_N(\mu)| \leq \frac{1}{\beta_{LB}} \|r_N(\mu)\|_{\mathcal{Y}'}\, |||\tilde{r}_{\tilde{N}}(\mu)|||_{\mathcal{X}', \delta}$.

Proof. Since $\beta^*_{\mathrm{LB}} = \beta_{\mathrm{LB}}$, we obtain

$$\beta_{\mathrm{LB}}\, \|\tilde{e}_{\tilde{N}}(\mu)\|_{\mathcal{Y}} \leq \sup_{w_\delta \in \mathcal{X}} \frac{b(w_\delta, \tilde{e}_{\tilde{N}}(\mu); \mu)}{|||w_\delta|||_{\mathcal{X}, \delta}} = \sup_{w_\delta \in \mathcal{X}} \frac{\tilde{r}_{\tilde{N}}(w_\delta; \mu)}{|||w_\delta|||_{\mathcal{X}, \delta}} = |||\tilde{r}_{\tilde{N}}(\mu)|||_{\mathcal{X}', \delta},$$

which proves (a). In order to show (b), we first note that

$$\begin{aligned} s_\delta(\mu) - s_N(\mu) &= J(e_N(\mu)) + r_N(z_{\tilde{N}}(\mu)) = J(e_N(\mu)) + b(e_N(\mu), z_{\tilde{N}}(\mu); \mu) \\ &= -\tilde{r}_{\tilde{N}}(e_N(\mu); \mu) \end{aligned}$$

so that $|s_\delta(\mu) - s_N(\mu)| \leq |||\tilde{r}_{\tilde{N}}(\mu)|||_{\mathcal{X}', \delta}\, \|e_N(\mu)\|_{\mathcal{X}, \delta}$ so that (b) follows by Proposition 3.1, (a). □

Remark 3.4. *Note that both estimates in Proposition 3.3 do not depend on the time T. Again, however, one expects that the space-time norms of the residuals will show T-dependence, which is due to the nature of the evolution problem.*

Let us comment on the numerical realization of (3.8). We are looking for $z_\delta(\mu) = \sum_{\ell=1}^{K} \sum_{j=1}^{n_h} z_j^\ell\, \tau^\ell \otimes \phi_j \in \mathcal{Y}_\delta$, $\mathbf{z}_\delta^\ell := (z_j^\ell)_{j=1,\ldots,n_h}$. Then, for $1 \leq i \leq n_h$, we obtain

$$\begin{aligned} b(\sigma^K \otimes \phi_i) &= \sum_{\ell=1}^{K} \sum_{j=1}^{n_h} z_j^\ell \left[(\dot{\sigma}^K, \tau^\ell)_{L_2(I)}\, (\phi_i, \phi_j)_H + (\sigma^K, \tau^\ell)_{L_2(I)}\, a(\phi_i, \phi_j) \right] \\ &= \sum_{j=1}^{n_h} \left(z_j^K\, (\phi_i, \phi_j)_H + \frac{\Delta t}{2} z_j^K\, a(\phi_i, \phi_j) \right) = [(\mathbf{M}_h^{\text{space}} + \frac{\Delta t}{2} \mathbf{A}_h^{\text{space}}) \mathbf{z}^K]_i \end{aligned}$$

and $J(\sigma^K \otimes \phi_i) = \frac{\Delta t}{2} \ell(\phi_i)$, so that $\mathbf{z}_\delta^K(\mu)$ can be computed via the solution of

$$(\mathbf{M}_h^{\text{space}} + \frac{\Delta t}{2} \mathbf{A}_h^{\text{space}}) \mathbf{z}_\delta^K(\mu) = -\frac{\Delta t}{2} \mathbf{l}, \tag{3.10}$$

where $\mathbf{l} := (\ell(\phi_i))_{i=1,\dots,n_h}$. Correspondingly, we obtain for $k = K-1,\dots,1$

$$\begin{aligned} b(\sigma^k \otimes \phi_i) &= \sum_{\ell=1}^{K}\sum_{j=1}^{n_h} z_j^\ell \,[(\dot\sigma^k, \tau^\ell)_{L_2(I)}\,(\phi_i,\phi_j)_H + (\sigma^k,\tau^\ell)_{L_2(I)}\,a(\phi_i,\phi_j)] \\ &= \sum_{j=1}^{n_h}[(z_k^k - z_j^{k+1})(\phi_i,\phi_j)_H + \frac{\Delta t}{2}(z_j^k + z_j^{k+1})\,a(\phi_i,\phi_j)] \\ &= [\mathbf{M}_h^{\text{space}}(\mathbf{z}_\delta^k(\mu) - \mathbf{z}_\delta^{k+1}(\mu)) + \frac{\Delta t}{2}\mathbf{A}_h^{\text{space}}(\mathbf{z}_\delta^k(\mu) + \mathbf{z}_\delta^{k+1}(\mu))]_i \end{aligned}$$

as well as $J(\sigma^k \otimes \phi_i) = \Delta t\,\ell(\phi_i)$, so that for $k = K-1,\dots,1$

$$(\mathbf{M}_h^{\text{space}} + \frac{\Delta t}{2}\mathbf{A}_h^{\text{space}})\mathbf{z}_\delta^k(\mu) = -\Delta t\,\mathbf{l} + (\mathbf{M}_h^{\text{space}} - \frac{\Delta t}{2}\mathbf{A}_h^{\text{space}})\mathbf{z}_\delta^{k+1}(\mu). \tag{3.11}$$

This means that (3.10) and (3.11) is an iterative procedure for computing the dual truth solution very similar to a backward Crank-Nicholson scheme. We do not need to solve a coupled space-time problem.

3.3. **Numerical realization.**

We are now going to consider the quantities that we have to determine while numerically approximating terms like the inf-sup-constants.

Norms. Let $w_\delta = \sum_{i=1}^r \sum_{k=1}^{n_h} w_k^i\,\sigma^i \otimes \phi_k \in \mathcal{X}_\delta$, $\mathbf{w}_\delta := (w_k^i)_{i,k}$. Then,

$$\begin{aligned} \|w_\delta\|^2_{L_2(I;V)} &= \int_I \|w_\delta(t)\|_V^2\,dt = \sum_{k,\ell=1}^{K}\sum_{i,j=1}^{n_h} w_k^i w_\ell^j \int_I \sigma^k(t)\,\sigma^\ell(t)\,(\phi_i,\phi_j)_V\,dt \\ &= \mathbf{w}_\delta^T(\mathbf{M}_{\Delta t}^{\text{time}} \otimes \mathbf{V}_h^{\text{space}})\mathbf{w}_\delta, \end{aligned}$$

where $\mathbf{M}_{\Delta t}^{\text{time}}$ is the temporal mass matrix and $\mathbf{V}_h^{\text{space}} = [(\phi_k,\phi_l)_V]_{k,l}$ the spatial matrix w.r.t. the V-inner product. For the discrete norm $|||\cdot|||_{\mathcal{X},\delta}$, we need $|||\bar w_\delta|||_{L_2(I;V)}$. We obtain

$$\begin{aligned} \|\bar w_\delta\|^2_{L_2(I;V)} &= \sum_{k=1}^{K}\int_{I^k}(\bar w^k(t), \bar w^k(t))_V\,dt = \frac{1}{\Delta t}\sum_{k=1}^{K}\Big(\int_{I^k} w(t)\,dt, \int_{I^k} w(s)\,ds\Big)_V \\ &= \Delta t\sum_{k=1}^{K}\sum_{i,j=1}^{n_h}\overline{w}_i^k\overline{w}_j^k(\phi_i,\phi_j)_V = \mathbf{w}_\delta^T(\overline{\mathbf{M}}_{\Delta t}^{\text{time}} \otimes \mathbf{V}_h^{\text{space}})\mathbf{w}_\delta, \end{aligned}$$

where $\overline{\mathbf{M}}_{\Delta t}^{\text{time}} := \Delta t\,\text{tridiag}(1/4, 1/2, 1/4)$.

The second part of the $\mathcal{X}$-norm, $\|\dot w_\delta\|_{L_2(I;V')}$, is a little bit more involved due to the appearance of the V'-norm. Given $\tilde v_h = \sum_{k=1}^{n_h}\tilde v_k\,\phi_k \in V_h$, $\tilde{\mathbf{v}}_h = (\tilde v_k)_k$, we need the Riesz representation $\hat{\tilde v}_h = \sum_{k'=1}^{n_h}\hat{\tilde v}_{k'}\,\phi_{k'}$, $\hat{\tilde{\mathbf{v}}}_h = (\hat{\tilde v}_{k'})_{k'}$ (since we know from the Riesz representation theorem that $\|\hat{\tilde v}_h\|_V = \|\tilde v_h\|_{V'}$), which is determined by the condition

$$(\hat{\tilde v}_h, \phi_\ell)_V \equiv \sum_{k'=1}^{n_h}\hat{\tilde v}_{k'}(\phi_{k'},\phi_\ell)_V = \sum_{k=1}^{n_h}\tilde v_k(\phi_k,\phi_\ell)_H \equiv (\tilde v_h,\phi_\ell)_H \quad \forall \ell = 1,\dots,n_h,$$

or in condensed form $\mathbf{V}_h^{\text{space}}\hat{\tilde{\mathbf{v}}}_h = \mathbf{M}_h^{\text{space}}\tilde{\mathbf{v}}_h$, i.e., $\hat{\tilde{\mathbf{v}}}_h = (\mathbf{V}_h^{\text{space}})^{-1}\mathbf{M}_h^{\text{space}}\tilde{\mathbf{v}}_h$ for the coefficients. Then,

$$\begin{aligned} \|\tilde{v}_h\|_{V'}^2 &= \|\hat{\tilde{v}}_h\|_V^2 = \hat{\tilde{\mathbf{v}}}_h^T \mathbf{V}_h^{\text{space}} \hat{\tilde{\mathbf{v}}}_h \\ &= ((\mathbf{V}_h^{\text{space}})^{-1}\mathbf{M}_h^{\text{space}}\tilde{\mathbf{v}}_h)^T\, \mathbf{V}_h^{\text{space}}(\mathbf{V}_h^{\text{space}})^{-1}\mathbf{M}_h^{\text{space}}\tilde{\mathbf{v}}_h \\ &= \tilde{\mathbf{v}}_h^T\, \mathbf{M}_h^{\text{space}}(\mathbf{V}_h^{\text{space}})^{-1}\mathbf{M}_h^{\text{space}}\, \tilde{\mathbf{v}}_h. \end{aligned}$$

Using this, we get

$$\begin{aligned} \|\dot{w}_\delta\|_{L_2(I;V')}^2 &= \sum_{k,\ell=1}^{K}\sum_{i,j=1}^{n_h} w_i^k w_j^\ell \int_I \dot{\sigma}^k(t)\,\dot{\sigma}^\ell(t)(\mathbf{M}_h^{\text{space}}(\mathbf{V}_h^{\text{space}})^{-1}\mathbf{M}_h^{\text{space}})_{i,j}\,dt \\ &= \mathbf{w}_\delta^T\big(\mathbf{V}_{\Delta t}^{\text{time}} \otimes (\mathbf{M}_h^{\text{space}}(\mathbf{V}_h^{\text{space}})^{-1}\mathbf{M}_h^{\text{space}})\big)\mathbf{w}_\delta, \end{aligned}$$

where $\mathbf{V}_{\Delta t}^{\text{time}} = [(\dot{\sigma}^k, \dot{\sigma}^\ell)_{L_2(I)}]_{k,\ell}$ is the temporal matrix of the derivatives. As for the last part, we obtain by $\sigma^k(T) = \delta_{k,K}$

$$\|w_\delta(T)\|_H^2 = \sum_{i,j=1}^{n_h} w_i^K\, w_j^K\, (\phi_i, \phi_j)_H = (\mathbf{w}_\delta^K)^T \mathbf{M}_h^{\text{space}} \mathbf{w}_\delta^K.$$

Consequently, we obtain for the norm $\|w_\delta\|_{\mathcal{X}}^2 = \mathbf{w}_\delta^T \mathbf{X}_\delta \mathbf{w}_\delta + (\mathbf{w}_\delta^K)^T \mathbf{V}_h^{\text{space}} \mathbf{w}_\delta^K$ with

$$\mathbf{X}_\delta := \mathbf{M}_{\Delta t}^{\text{time}} \otimes \mathbf{V}_h^{\text{space}} + \mathbf{V}_{\Delta t}^{\text{time}} \otimes (\mathbf{M}_h^{\text{space}}(\mathbf{V}_h^{\text{space}})^{-1}\mathbf{M}_h^{\text{space}}). \tag{3.12}$$

For the discrete norm, we just need to modify $\mathbf{X}_\delta$ to $\mathbf{X}_\delta^{|||\cdot|||} := \overline{\mathbf{M}}_{\Delta t}^{\text{time}} \otimes \mathbf{V}_h^{\text{space}} + \mathbf{V}_{\Delta t}^{\text{time}} \otimes (\mathbf{M}_h^{\text{space}}(\mathbf{V}_h^{\text{space}})^{-1}\mathbf{M}_h^{\text{space}})$.

For $v_\delta = \sum_{k=1}^K \sum_{i=1}^{n_h} v_i^k\, \tau^k \otimes \phi_i \in \mathcal{Y}_h$ we can use very similar arguments and get $\|v_\delta\|_{\mathcal{Y}}^2 = \mathbf{v}_\delta^T \mathbf{Y}_\delta \mathbf{v}_\delta$ with

$$\mathbf{Y}_\delta := \mathbf{G}_{\Delta t}^{\text{time}} \otimes \mathbf{V}_h^{\text{space}} \tag{3.13}$$

and $\mathbf{G}_{\Delta t}^{\text{time}} = [(\tau^k, \tau^\ell)_{L_2(I)}]_{k,\ell}$ being the mass matrix of the $Q_{\Delta t}$-basis functions. In our case of piecewise constants, this coincides with $\Delta t\ \mathbf{I}_{\Delta t}^{\text{time}}$.

Bilinear form. We have already seen that $b(w_\delta, v_\delta) = \mathbf{w}_\delta^T \mathbf{B}_\delta \mathbf{v}_\delta$ with $\mathbf{B}_\delta$ given by (2.14).

Supremizing operator. Finally, we determine the supremizing operator for the bilinear form b, i.e., $T_\delta w_\delta = \arg\sup_{v_\delta \in \mathcal{Y}_\delta} \frac{b(w_\delta, v_\delta)}{\|v_\delta\|_{\mathcal{Y}}}$ for given $w_\delta \in \mathcal{X}_\delta$. It is well-known that $T_\delta w_\delta \in \mathcal{Y}_\delta$ is the solution of $(T_\delta w_\delta, v_\delta)_{\mathcal{Y}} = b(w_\delta, v_\delta)$ for all $v_\delta \in \mathcal{Y}_\delta$. The coefficients $\mathbf{t}_\delta$ of $T_\delta w_\delta$ are then given by $\mathbf{t}_\delta = \mathbf{Y}_\delta^{-1}\mathbf{B}_\delta^T \mathbf{w}_\delta$. Finally, it is also well known that

$$\beta_\delta = \inf_{w_\delta \in \mathcal{X}_\delta} \frac{\|T w_\delta\|_{\mathcal{Y}}}{\|w_\delta\|_{\mathcal{X}}}$$

and we get

$$\frac{\|T_\delta w_\delta\|_{\mathcal{Y}}^2}{\|w_\delta\|_{\mathcal{X}}^2} = \frac{\mathbf{t}_\delta^T \mathbf{Y}_\delta \mathbf{t}_\delta}{\mathbf{w}_\delta^T \mathbf{X}_\delta \mathbf{w}_\delta + (\mathbf{w}_\delta^K)^T \mathbf{M}_h^{\text{space}} \mathbf{w}_\delta^K} = \frac{\mathbf{w}_\delta^T \mathbf{B}_\delta \mathbf{Y}_\delta^{-1} \mathbf{B}_\delta^T \mathbf{w}_\delta}{\mathbf{w}_\delta^T \mathbf{X}_\delta \mathbf{w}_\delta + (\mathbf{w}_\delta^K)^T \mathbf{M}_h^{\text{space}} \mathbf{w}_\delta^K}$$

with the involved matrices defined in (3.12), (3.13) and (2.14). Thus, we need to determine the square root of the smallest eigenvalue of the generalized eigenvalue problem $\mathbf{B}_\delta \mathbf{Y}_\delta^{-1} \mathbf{B}_\delta^T \mathbf{v} = \eta\, \mathbf{X}_\delta \mathbf{v}$.

Error estimators. Since the computation of lower bounds for the inf-sup parameters has already been described, it remains to detail numerical schemes for the dual norms of the residuals, i.e., $\|r_N(\mu)\|_{\mathcal{Y}'}$ and $|||\tilde{r}_{\tilde{N}}(\mu)|||_{\mathcal{X}',\delta}$. We have already seen that $\|r_N(\mu)\|_{\mathcal{Y}'} = \|\hat{r}_N(\mu)\|_{\mathcal{Y}}$ with the Riesz representation $\hat{r}_N(\mu) \in \mathcal{Y}_\delta$ which is given by $(\hat{r}_N(\mu), v_\delta)_{\mathcal{Y}} = f(v_\delta;\mu) - b(u_N(\mu), v_\delta;\mu)$ for all $v_\delta \in \mathcal{Y}_\delta$. In matrix-vector form for the coefficients this reads

$$\mathbf{Y}_\delta \hat{\mathbf{r}}_N(\mu) = \mathbf{f}_\delta(\mu) - \mathbf{B}_\delta^T \mathbf{u}_N(\mu),$$

where as above $\mathbf{Y}_\delta = \mathbf{G}_{\Delta t}^{\text{time}} \otimes \mathbf{V}_h^{\text{space}}$, $\mathbf{f}_\delta(\mu) = (f(\sigma^k \otimes \phi_i;\mu))_{k=1,\ldots,K;\ i=1,\ldots,n_h}$, $\mathbf{B}_\delta = \mathbf{N}_{\Delta t}^{\text{time}} \otimes \mathbf{M}_h^{\text{space}} + \mathbf{M}_{\Delta t}^{\text{time}} \otimes \mathbf{A}_h^{\text{space}}$ and $\mathbf{u}_N(\mu)$ being the vector of expansion coefficients of the RB-solution. Finally, for the right-hand side using the affine assumption (3.2) and defining $q_G \in \mathcal{Y}'$ by $[g_q, v]_{\mathcal{H}} = f_q(v)$, $v \in \mathcal{Y}$, we get

$$\begin{aligned} f(\sigma^k \otimes \phi_i;\mu) &= \sum_{q=1}^{Q} \theta_q^f(\mu)\, f_q(\sigma^k \otimes \phi_i) = \sum_{q=1}^{Q} \theta_q^f(\mu)\, [g_q, \sigma^k \otimes \phi_i]_{\mathcal{H}} \\ &= \sum_{k=1}^{K} \sum_{q=1}^{Q} \theta_q^f(\mu) \langle g_q(t^k), \phi_i \rangle_{V' \times V}, \end{aligned}$$

where we used the fact that σ^k are piecewise linear and are thus integrated exactly by a trapezoidal rule. This shows that expanding $g_q(t^k)$ in any appropriate basis gives rise to a tensor-product representation of $\mathbf{f}_\delta(\mu)$. Hence, the Riesz representation calculation is reduced to a sequence of K uncoupled spatial problems in V — just as in the non-space-time case.

The situation is different for $|||\tilde{r}_{\tilde{N}}(\mu)|||_{\mathcal{X}',\delta} = |||\hat{\tilde{r}}_{\tilde{N}}(\mu)|||_{\mathcal{X},\delta}$, where the Riesz representation $\hat{\tilde{r}}_{\tilde{N}} \in \mathcal{X}_\delta$ is defined by $(\hat{\tilde{r}}(\mu), w_\delta)_{\mathcal{X},\delta} = -J(w_\delta) - b(w_\delta, z_{\tilde{N}}(\mu);\mu)$ and the truth inner product is defined as

$$(v_\delta, w_\delta)_{\mathcal{X},\delta} := (\dot{v}_\delta, \dot{w}_\delta)_{V'} + (\bar{v}_\delta, \bar{w}_\delta)_V + (v_\delta(T), w_\delta(T))_H, \quad v_\delta, w_\delta \in \mathcal{X}_\delta.$$

In this case too, though less obviously, it is also possible to calculate the dual norm as a sequence of uncoupled spatial problems — but now we require both a forward and a backward sweep, for a total of $3K$ spatial problems [16].

4. Numerical Results

Now, let $\mu = (\mu_1, \mu_2) \in \mathcal{D} := \mathbb{R}^2$ be a parameter vector and $A = A(\mu) := -\Delta u + \mu_1\, \boldsymbol{\beta}(x) \cdot \nabla u + \mu_2\, u$, i.e., a diffusion-convection-reaction operator with convection field $\boldsymbol{\beta}$. We report numerical results for the Crank–Nicolson scheme for various choices of the parameters μ_1, μ_2 as well as for different time steps Δt and uniform mesh sizes h. For simplicity, we consider the univariate case (in space) $\Omega = (0,1)$ and choose $\boldsymbol{\beta}(x) = x - \frac{1}{2}$. Let us denote by $\beta_\delta(\mu;T)$, $\gamma_\delta(\mu;T)$ the numerical values for the truth inf-sup and continuity constants, respectively, corresponding to parameter μ and final time T. All computations are based upon solving corresponding eigenproblems, which correspond to homogeneous initial conditions and right-hand sides.

We start by confirming Proposition 2.9. Thus, we choose $\mu_1 = \mu_2 = 0$; for several values of T, h, and Δt we invariantly obtain 1.000 for both $\beta_\delta(\mu;T)$ and $\gamma_\delta(\mu;T)$, as must be the case.

The next issue is that we want to confirm the independence of $\beta_\delta(\mu;T)$ with respect to the discretization parameters $\delta = (\Delta t, h)$. In Table 1 we consider the

N_t; N_s	9	14	19	24	29
10	5.7242e-02	5.8419e-02	5.8863e-02	5.9073e-02	5.9188e-02
15	5.7459e-02	5.8631e-02	5.9072e-02	5.9281e-02	5.9395e-02
20	5.7535e-02	5.8704e-02	5.9145e-02	5.9353e-02	5.9467e-02
25	5.7570e-02	5.8739e-02	5.9179e-02	5.9387e-02	5.9501e-02
30	5.7589e-02	5.8757e-02	5.9197e-02	5.9405e-02	5.9519e-02
35	5.7600e-02	5.8768e-02	5.9208e-02	5.9416e-02	5.9530e-02
40	5.7608e-02	5.8775e-02	5.9216e-02	5.9423e-02	5.9537e-02

Table 1. Long time-behavior of the inf-sup constant $\beta_\delta((50,10);0.2)$ for various choices of $\delta = (\frac{1}{N_s}, \frac{0.2}{N_t})$.

		β_δ		
N_t	T	$\mu_1 = 50$	$\mu_1 = 100$	$\mu_1 = 150$
10	0.200000	2.081838e-01	9.189784e-02	5.605419e-02
20	0.400000	1.164954e-01	4.767668e-02	2.858245e-02
30	0.600000	8.062734e-02	3.200346e-02	1.911024e-02
40	0.800000	6.187347e-02	2.405788e-02	1.434315e-02
50	1.000000	5.040255e-02	1.926570e-02	1.147687e-02
60	1.200000	4.267737e-02	1.606301e-02	9.564429e-03
70	1.400000	3.712638e-02	1.377228e-02	8.197915e-03
80	1.600000	3.294756e-02	1.205285e-02	7.172878e-03
90	1.800000	2.968954e-02	1.071484e-02	6.375585e-03
100	2.000000	2.707910e-02	9.644058e-03	5.737750e-03

Table 2. Long time-behavior of the inf-sup constant in the convection case $\mu = (\mu_1, 0)$.

case $\mu = (50, 10)$ with the final time $T = 0.2$. We clearly see the rapid convergence for $\Delta t \to 0$ as well as for $h \to 0$. This behavior has been observed for various choices of the parameters and final time.

Next, we investigate the case of convection, $\mu_2 = 0$, in which case a is coercive only for $\mu_1 < 2\pi^2$. We are particularly interested in the long-time behavior. The results are displayed in Table 2 for the choice $N_s = 19$ and $N_t = 10$ per time interval of length 0.2. The displayed numbers, however, are relatively invariant for sufficiently small h and Δt. We observe numerically an overall behavior of $\beta_\delta((\mu_1, 0); T) \sim (\mu_1 T)^{-1}$ and $\gamma_\delta((\mu_1, 0); T) \sim \mu_1$ (the latter is readily proven, but not the former). Note $T = \mathcal{O}(1)$ is effectively a "long time" in convective units, $1/\mu_1$. We emphasize that although the problem is non-coercive, the problem is asymptotically stable in the sense that all eigenvalues η of $-a(\psi, \phi) = \eta\langle\psi, \phi\rangle_{V' \times V}$ lie in the left-hand plane; this stability is reflected in the inf-sup behavior. In contrast, a standard energy approach [5] gives effective inf-sup constants on the order of $e^{-\mu_1 T}$ (here about 10^{-8}). Hence, the traditional method fails to provide useful results, whereas our new approach, which reflects the true time-coupled properties of the system, yields relatively sharp error bounds.

Finally, we consider the case $\mu_1 = 0$ which gives rise to an asymptotically unstable (and non-coercive) system for $\mu_2 < -\pi^2$. This means that any error estimate

		β_δ			
N_t	T	$N_s = 19$	$N_s = 24$	$N_s = 29$	$N_s = 34$
10	0.200000	1.328157e-01	1.327088e-01	1.326507e-01	1.326157e-01
20	0.400000	1.747513e-02	1.743612e-02	1.741498e-02	1.740224e-02
30	0.600000	2.297580e-03	2.289068e-03	2.284460e-03	2.281686e-03
40	0.800000	3.020714e-04	3.005078e-04	2.996622e-04	2.991535e-04
50	1.000000	3.971441e-05	3.945054e-05	3.930789e-05	3.922218e-05

Table 3. Long time-behavior of the inf-sup constant in the asymptotically unstable case $\mu = (0, -20)$ for different spatial resolution.

must grow exponentially with the final time T. We observe this for our estimator as well, as Table 3 shows, the values are in the order of $e^{\mu_2 T}$.

5. Conclusions

We have introduced new a posteriori error bounds based upon a space-time Petrov-Galerkin discretization of linear parabolic partial differential equations. This allows to use standard estimates for the error in terms of the dual norm of the residual multiplied with the inverse of the inf-sup constant. We have shown that the discrete inf-sup constant is quite favorable, in particular for long-time integration. In the interim, this approach has been extended to the Burger's equation [17] and to the Boussinesq equations [16], again resulting in significant quantitative improvements for the error estimates compared to earlier approaches [5].

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