

Adaptive Port Reduction in Static Condensation [★]

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Abstract: We introduce a framework for adaptive reduction of the degrees of freedom associated with ports in static condensation (SC). We apply this framework to the SC reduced basis (RB) method and thus combine parametric model order reduction for the interior of a component with model order reduction on the ports in order to rapidly construct an approximate Schur complement linear system of reduced size. The port reduction framework invokes quasi-rigorous *a posteriori* error bounds for adaptation and allows a combination of *empirical* functions (snapshot-based) and eigenfunctions for the representation of the solution on the ports.

Keywords: static condensation; reduced basis method; model order reduction; port reduction; a posteriori error estimation

1. INTRODUCTION

The static condensation reduced basis (SCRB) method (Huynh et al. (2011)) provides a computational framework for the solution of parameter dependent (linear elliptic) partial differential equations (PDEs) where the spatial domain consists of a number of components interconnected through predefined ports to form a global system. Associated with each component is a set of scalar parameters that describe properties such as materials or geometry. The methodology introduced in Huynh et al. (2011) provides rapid solution of the global PDE for different parameter values as well as for different topological configurations of the system. Moreover, it allows treatment of parametrized PDEs with many parameters since the RB approximations are local to each component.

In a standard SC framework it is necessary to solve (with a finite element (FE) method, say) a PDE local to each component for each of the degrees of freedom associated with the ports in order to assemble the Schur complement system. Within the SCRB framework, each of these local solves is replaced by an RB approximation (Rozza et al. (2008)) specifically tailored to the *local* parameter dependence associated with the component. As a result, an approximate Schur complement system can be rapidly constructed and solved for any given *global* parameter value.

In this paper, we introduce a framework for adaptive reduction of the degrees of freedom associated with the ports. We thus combine model order reduction for the

interior of a component with port reduction in order to rapidly construct a Schur complement linear system of reduced size. The port reduction framework allows a combination of *empirical* functions — functions informed by the solution at the ports for selected values of the global parameter — and more general eigenfunctions for the representation of the solution on a port. Port reduction is considered in earlier works; we refer to Jakobsson et al. (2011); Hetmaniuk and Lehoucq (2010) in which an eigenfunction expansion (of the Laplacian) is used in the context of more classical component mode synthesis. We here apply a similar but more general technique, and we also provide a quasi-rigorous¹ and inexpensive *a posteriori* bound for the error between the port-reduced SCRB approximation and the underlying FE approximation.

In the next section we briefly review the SCRB framework from Huynh et al. (2011). In Section 3 we introduce the new port reduction framework and in Section 4 we conclude with numerical results.

2. REDUCED BASIS STATIC CONDENSATION

2.1 Approximation

We introduce a parameter domain $\mathcal{D} \subset \mathbb{R}^P$ and a spatial domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$; a particular parameter value shall be denoted as $\mu \in \mathcal{D}$. We then introduce a Hilbert space $X(\Omega)$, $H_0^1(\Omega) \subset X(\Omega) \subset H^1(\Omega)$; we let $X^{\mathcal{N}}(\Omega) \subset X(\Omega)$ denote an FE discretization of $X(\Omega)$ where $\mathcal{N} = \dim(X^{\mathcal{N}}(\Omega))$. We further introduce a continuous and symmetric parametrized bilinear form $a : X \times$

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¹ Rigorous under an eigenvalue proximity assumption; see (19) and Footnote 3.

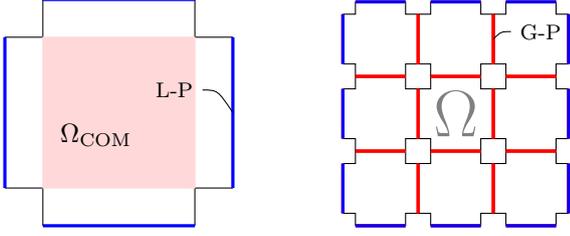


Fig. 1. Individual components interconnect through ports to form a global system.

$X \times \mathcal{D} \rightarrow \mathbb{R}$ and a bounded parametrized linear functional $f : X \times \mathcal{D} \rightarrow \mathbb{R}$. We may then introduce our FE-discretized parameter dependent elliptic PDE as follows: given any $\mu \in \mathcal{D}$, find $u^\mathcal{N}(\mu) \in X^\mathcal{N}(\Omega)$ such that

$$a(u^\mathcal{N}(\mu), v; \mu) = f(v; \mu), \quad \forall v \in X^\mathcal{N}(\Omega); \quad (1)$$

we assume for our computational (offline-online) procedures (Rozza et al. (2008)) that a and f admit affine separations of function and parameter dependence as

$$a(w, v; \mu) = \sum_{q=1}^{Q_a} a^q(w, v) \Theta_a^q(\mu) \quad (2)$$

and

$$f(v; \mu) = \sum_{q=1}^{Q_f} f^q(v) \Theta_f^q(\mu), \quad (3)$$

where the a^q and f^q are parameter-independent forms, the Θ_a^q and Θ_f^q are parameter-dependent functions, and $Q_a \leq Q$ and $Q_f \leq Q$ for Q finite.

We consider a decomposition of the global domain Ω into domains Ω_{COM} associated with each *component* $\text{COM} \in \mathcal{C}_{\text{SYS}}$ as

$$\bar{\Omega} = \bigcup_{\text{COM} \in \mathcal{C}_{\text{SYS}}} \bar{\Omega}_{\text{COM}}; \quad (4)$$

here \mathcal{C}_{SYS} denotes the set of components of which the global system is composed. Each component has a set of local ports (L-P) \mathcal{P}_{COM} . A single local port on any $\text{COM} \in \mathcal{C}_{\text{SYS}}$ may form a global port (G-P), or two local ports from different $\text{COM} \in \mathcal{C}_{\text{SYS}}$ may form a global port; let \mathcal{P}_{SYS} denote the set of global ports. The map \mathcal{G} from (L-P, COM) to G-P thus defines the particular configuration of the global system. We provide an illustration in Figure 1: nine components of the same type, each of which has four local ports (blue), form a global system with 24 global ports. Note that in the case that we impose only Dirichlet external boundary conditions, there is only twelve global ports (indicated in red).

For each $\text{G-P} \in \mathcal{P}_{\text{SYS}}$ let $\Gamma_{\text{G-P}}$ denote the associated spatial domain; we introduce the FE space $X^\mathcal{N}(\Gamma_{\text{G-P}})$ as the restriction of $X^\mathcal{N}(\Omega)$ to $\Gamma_{\text{G-P}}$; let $n_{\text{G-P}} \equiv \dim(X^\mathcal{N}(\Gamma_{\text{G-P}}))$. We represent the $n_{\text{G-P}}$ degrees of freedom associated with G-P in terms of functions $\chi_{1, \text{G-P}}, \dots, \chi_{n_{\text{G-P}}, \text{G-P}} \in X^\mathcal{N}(\Gamma_{\text{G-P}})$, the particular choice for which is the main focus of this paper. We also introduce $n_{\text{SC}} = \sum_{\text{G-P}} n_{\text{G-P}}$.

For each $\text{COM} \in \mathcal{C}_{\text{SYS}}$ we introduce a space $X_{\text{COM};0}^\mathcal{N}$ as the restriction of $X^\mathcal{N}(\Omega)$ to Ω_{COM} with homogeneous Dirichlet boundary conditions on the local ports. We then introduce

a function $b_{f, \text{COM}}^\mathcal{N}(\mu) \in X_{\text{COM};0}^\mathcal{N}$ such that, for any $\mu \in \mathcal{D}$,²

$$a(b_{f, \text{COM}}^\mathcal{N}(\mu), v; \mu) = f(v; \mu), \quad \forall v \in X_{\text{COM};0}^\mathcal{N}; \quad (5)$$

and, for each L-P $\in \mathcal{P}_{\text{COM}}$, we introduce $n_{\mathcal{G}(\text{L-P}, \text{COM})}$ functions $b_{k, \text{L-P}, \text{COM}}^\mathcal{N}(\mu) \in X_{\text{COM};0}^\mathcal{N}$, $1 \leq k \leq n_{\mathcal{G}(\text{L-P}, \text{COM})}$, such that, for any $\mu \in \mathcal{D}$,

$$a(b_{k, \text{L-P}, \text{COM}}^\mathcal{N}(\mu), v; \mu) = -a(\psi_{k, \text{L-P}, \text{COM}}, v; \mu), \quad \forall v \in X_{\text{COM};0}^\mathcal{N}. \quad (6)$$

Here, $\psi_{k, \text{L-P}, \text{COM}}$ denotes a harmonic lift of the trace $\chi_{k, \mathcal{G}(\text{L-P}, \text{COM})} \in X^\mathcal{N}(\Gamma_{\mathcal{G}(\text{L-P}, \text{COM})})$ into the interior Ω_{COM} .

On each COM, we can write the solution to (1) as

$$u_{\text{COM}}^\mathcal{N}(\mu) = b_{\text{COM}}^\mathcal{N}(\mu) + \sum_{\text{L-P} \in \text{COM}} \sum_{k=1}^{n_{\mathcal{G}(\text{L-P}, \text{COM})}} U_{k, \mathcal{G}(\text{L-P}, \text{COM})}(\mu) \psi_{k, \text{L-P}, \text{COM}} \quad (7)$$

where

$$b_{\text{COM}}^\mathcal{N}(\mu) = b_{f, \text{COM}}^\mathcal{N}(\mu) + \sum_{\text{L-P} \in \text{COM}} \sum_{k=1}^{n_{\mathcal{G}(\text{L-P}, \text{COM})}} U_{k, \mathcal{G}(\text{L-P}, \text{COM})}(\mu) b_{k, \text{L-P}, \text{COM}}^\mathcal{N}(\mu) \quad (8)$$

and the $U_{k, \mathcal{G}(\text{L-P}, \text{COM})}$ are unknown coefficients associated with the degrees of freedom on the *global port* $\mathcal{G}(\text{L-P}, \text{COM})$. The global solution over Ω can then be written as

$$u^\mathcal{N}(\mu) = \sum_{\text{COM} \in \mathcal{C}_{\text{SYS}}} b_{f, \text{COM}}^\mathcal{N}(\mu) + \sum_{\text{G-P} \in \mathcal{P}_{\text{SYS}}} \sum_{k=1}^{n_{\text{G-P}}} U_{k, \text{G-P}}(\mu) \Phi_{k, \text{G-P}}^\mathcal{N}(\mu) \quad (9)$$

where

$$\begin{aligned} \Phi_{k, \text{G-P}}^\mathcal{N}(\mu) &\equiv \sum_{(\text{L-P}, \text{COM}) \in \mathcal{G}^{-1}(\text{G-P})} (b_{k, \text{L-P}, \text{COM}}^\mathcal{N}(\mu) + \psi_{k, \text{L-P}, \text{COM}}^\mathcal{N}). \end{aligned} \quad (10)$$

Insertion of (9) into (1) but restricted to

$$v \in \text{span}\{\Phi_{k, \text{G-P}}, 1 \leq k \leq n_{\text{G-P}}, \text{G-P} \in \mathcal{P}_{\text{SYS}}\} \quad (11)$$

leads to the Schur complement system

$$\mathbb{A}(\mu) \mathbb{U}(\mu) = \mathbb{F}(\mu) \quad (12)$$

of size $n_{\text{SC}} \times n_{\text{SC}}$ for the coefficients $U_{1, \text{G-P}}, \dots, U_{n_{\text{G-P}}, \text{G-P}}$, $\text{G-P} \in \mathcal{P}_{\text{SYS}}$.

Within the SCR method (Huynh et al. (2011)), the interior solve (5) required for each COM and the $n_{\mathcal{G}(\text{L-P}, \text{COM})}$ interior solves (6) required for each (L-P, COM) are replaced by RB solves (Rozza et al. (2008)) to obtain accurate and computationally efficient approximations $\tilde{b}_{f, \text{COM}}^\mathcal{N}(\mu) \approx b_{f, \text{COM}}^\mathcal{N}(\mu)$ and $\tilde{b}_{k, \text{L-P}, \text{COM}}^\mathcal{N}(\mu) \approx b_{k, \text{L-P}, \text{COM}}^\mathcal{N}(\mu)$ for all $\mu \in \mathcal{D}$. These approximations lead to an approximate Schur complement system

$$\tilde{\mathbb{A}}(\mu) \tilde{\mathbb{U}}(\mu) = \tilde{\mathbb{F}}(\mu) \quad (13)$$

for the coefficients $\tilde{U}_{1, \text{G-P}}, \dots, \tilde{U}_{n_{\text{G-P}}, \text{G-P}}$, $\text{G-P} \in \mathcal{P}_{\text{SYS}}$; we thus approximate $\tilde{\mathbb{U}}(\mu) \approx \tilde{\mathbb{U}}(\mu)$. We refer to Huynh et al. (2011) for a detailed derivation of (13).

² We extend all functions with subscript COM by zero outside Ω_{COM} and hence we may use the ‘‘global’’ forms a and f .

2.2 A Posteriori Error Estimation

It is shown in Huynh et al. (2011) that for symmetric elliptic parametrized PDEs

$$\begin{aligned} \|\mathbb{U}(\mu) - \widetilde{\mathbb{U}}(\mu)\|_2 &\leq \overline{\Delta\mathbb{U}}(\mu) \\ &\equiv \frac{\sigma_1(\mu) + \sigma_3(\mu) + \|\widetilde{\mathbb{F}}(\mu) - \widetilde{\mathbb{A}}(\mu)\widetilde{\mathbb{U}}(\mu)\|_2}{\widetilde{\lambda}_{\min}(\mu) - \sigma_2(\mu)}, \quad \forall \mu \in \mathcal{D}, \end{aligned} \quad (14)$$

where the $\sigma_i(\mu) \geq 0$, $i = 1, 2, 3$, relate to RB error bounds (Rozza et al. (2008)) and $\widetilde{\lambda}_{\min}(\mu)$ is the smallest eigenvalue of $\widetilde{\mathbb{A}}(\mu)$; we assume that $\sigma_2(\mu) < \widetilde{\lambda}_{\min}(\mu)$. In the case that (13) is solved exactly the residual term $\|\widetilde{\mathbb{F}}(\mu) - \widetilde{\mathbb{A}}(\mu)\widetilde{\mathbb{U}}(\mu)\|_2$ is zero.

The focus of this paper is port reduction and hence truncation of the degrees of freedom on the ports. Such a truncation corresponds to a perturbation of the system (13) and hence an inexact solution. If we separate the n_{SC} port degrees of freedom into n_A *active* degrees of freedom and n_I *inactive* degrees of freedom ($n_A + n_I = n_{SC}$), (13) can be written as

$$\begin{bmatrix} \widetilde{\mathbb{A}}_{AA}(\mu) & \widetilde{\mathbb{A}}_{AI}(\mu) \\ \widetilde{\mathbb{A}}_{IA}(\mu) & \widetilde{\mathbb{A}}_{II}(\mu) \end{bmatrix} \widetilde{\mathbb{U}}(\mu) = \begin{bmatrix} \widetilde{\mathbb{F}}_A(\mu) \\ \widetilde{\mathbb{F}}_I(\mu) \end{bmatrix}. \quad (15)$$

We then take the vector $\widehat{\mathbb{U}}(\mu) \equiv [\widetilde{\mathbb{U}}'(\mu), \overbrace{0, \dots, 0}^{n_I \text{ zeros}}]^T \in \mathbb{R}^{n_A + n_I}$ to approximate $\widetilde{\mathbb{U}}(\mu) \in \mathbb{R}^{n_{SC}}$, where

$$\widetilde{\mathbb{A}}_{AA}(\mu)\widetilde{\mathbb{U}}'(\mu) = \widetilde{\mathbb{F}}_A(\mu). \quad (16)$$

The error bound (14) is still valid for this approximation, but now with a non-zero residual computed as

$$\begin{aligned} R(\mu) &\equiv \|\widetilde{\mathbb{F}}(\mu) - \widetilde{\mathbb{A}}(\mu)\widehat{\mathbb{U}}(\mu)\|_2 \\ &= \|\widetilde{\mathbb{F}}_I(\mu) - \widetilde{\mathbb{A}}_{IA}(\mu)\widetilde{\mathbb{U}}'(\mu)\|_2 \end{aligned} \quad (17)$$

since the last n_I entries in $\widehat{\mathbb{U}}(\mu)$ are zero. Moreover since we do not assemble the full matrix $\widetilde{\mathbb{A}}(\mu)$ we can not compute $\widetilde{\lambda}_{\min}(\mu)$ directly; however, we can develop a lower bound for $\widetilde{\lambda}_{\min}(\mu)$ as follows (we omit here the μ -dependence for simplicity of notation).

Let $\{(\widetilde{\lambda}_i, \widetilde{v}_i), 1 \leq i \leq n_{SC}\}$ denote the eigenpairs associated with the SCRB eigenproblem $\widetilde{\mathbb{A}}\widetilde{v}_i = \widetilde{\lambda}_i\widetilde{v}_i$, and let $\{(\widetilde{\lambda}'_i, \widetilde{v}'_i), 1 \leq i \leq n_A\}$ denote the eigenpairs associated with the *active* eigenproblem

$$\widetilde{\mathbb{A}}_{AA}\widetilde{v}'_i = \widetilde{\lambda}'_i\widetilde{v}'_i; \quad (18)$$

we suppose that $\|\widetilde{v}_i\|_2 = \|\widetilde{v}'_j\|_2 = 1$, $1 \leq i \leq n_{SC}$, $1 \leq j \leq n_A$. Associate $\widetilde{\lambda}'_{\min}$ with the smallest eigenvalue of (18) and let $\widehat{v}_{\min} = [\widetilde{v}'_{\min}, 0, \dots, 0]^T$; we then express \widehat{v}_{\min} in terms of the \widetilde{v}_i as $\widehat{v}_{\min} \equiv \sum_{i=1}^{n_{SC}} \alpha_i \widetilde{v}_i$. Then, if $|\widetilde{\lambda}_{\min} - \widetilde{\lambda}'_{\min}| \leq |\widetilde{\lambda}_i - \widetilde{\lambda}'_{\min}|$ for $i = 2, \dots, n_{SC}$,³ it follows that (see also Isaacson and Keller (1994))

$$\begin{aligned} \|\widetilde{\mathbb{A}}\widehat{v}_{\min} - \widetilde{\lambda}'_{\min}\widehat{v}_{\min}\|_2^2 &= \left\| \sum_{i=1}^{n_{SC}} \alpha_i \widetilde{\mathbb{A}}\widetilde{v}_i - \widetilde{\lambda}'_{\min} \sum_{i=1}^{n_{SC}} \alpha_i \widetilde{v}_i \right\|_2^2 \\ &= \left\| \sum_{i=1}^{n_{SC}} \alpha_i (\widetilde{\lambda}_i - \widetilde{\lambda}'_{\min}) \widetilde{v}_i \right\|_2^2 = \sum_{i=1}^{n_{SC}} |\widetilde{\lambda}_i - \widetilde{\lambda}'_{\min}|^2 \|\alpha_i \widetilde{v}_i\|_2^2 \\ &\geq |\widetilde{\lambda}_{\min} - \widetilde{\lambda}'_{\min}|^2 \sum_{i=1}^{n_{SC}} \|\alpha_i \widetilde{v}_i\|_2^2 = |\widetilde{\lambda}_{\min} - \widetilde{\lambda}'_{\min}|^2 \end{aligned} \quad (19)$$

where orthogonality of the \widetilde{v}_i is used in the third and fifth steps. Hence

$$|\widetilde{\lambda}_{\min} - \widetilde{\lambda}'_{\min}| \leq \|\widetilde{\mathbb{A}}\widehat{v}_{\min} - \widetilde{\lambda}'_{\min}\widehat{v}_{\min}\|_2, \quad (20)$$

and a lower bound for $\widetilde{\lambda}_{\min}$ is

$$\begin{aligned} \widetilde{\lambda}_{\min, \text{LB}} &\equiv \widetilde{\lambda}'_{\min} - \|\widetilde{\mathbb{A}}\widehat{v}_{\min} - \widetilde{\lambda}'_{\min}\widehat{v}_{\min}\|_2 \\ &= \widetilde{\lambda}'_{\min} - \|\widetilde{\mathbb{A}}_{IA}\widetilde{v}'_{\min}\|_2, \end{aligned} \quad (21)$$

where the last equality follows since \widehat{v}_{\min} is zero for the last n_I entries and $(\widetilde{\lambda}'_{\min}, \widetilde{v}'_{\min})$ satisfy (18).

We thus see that we may obtain a quasi-rigorous *a posteriori* error bound with the computation of $R(\mu)$ from (17) and $\widetilde{\lambda}_{\min, \text{LB}}$ from (21) if we perform the relatively inexpensive additional assembly of $\widetilde{\mathbb{A}}_{IA}(\mu)$; we henceforth refer to this bound as $\widehat{\Delta\mathbb{U}}(\mu)$.

Remark 1. The error bound $\widehat{\Delta\mathbb{U}}(\mu)$ above requires the assembly of $\widetilde{\mathbb{A}}_{IA}(\mu)$ and hence the computational cost depends on n_I . As an alternative we may obtain a much less expensive error estimator if we assume that only the first $r|\mathcal{P}_{\text{SYS}}| \ll n_I$ few inactive degrees of freedom (r on each port) accommodate adequate approximations for $R(\mu)$ and $\widetilde{\lambda}_{\min, \text{LB}}(\mu)$. We may compute these approximations by replacing $\widetilde{\mathbb{A}}_{IA}$ above by a smaller matrix of size $r|\mathcal{P}_{\text{SYS}}| \times n_A$. The efficacy of this approach relies on a sensibly ordered modal representation of the degrees of freedom and furthermore on rapid decay in the Schur complement solution coefficients. We denote by $\widehat{\Delta\mathbb{U}}_r(\mu)$ the error estimator obtained with this approach.

2.3 Computational Procedures

The SCRB framework admits an offline-online computational decoupling. In the *offline* stage the RB approximation spaces for the interior solves on each $\text{COM} \in \mathcal{C}_{\text{SYS}}$ are constructed, and parameter independent quantities required for the subsequent online stage are computed and stored. This stage is computationally expensive since the training of the RB spaces involves \mathcal{N} -dependent FE solves *and* since we use a different RB model for each port degree-of-freedom.

In the *online* stage, given any $\mu \in \mathcal{D}$, first the RB approximations to (5) and (6) are computed together with RB *a posteriori* error bounds. Subsequently the global SCRB system (13) (or (16) in the case of port reduction) is assembled and solved, and the global *a posteriori* error bound (or estimator, c.f. Remark 1) is computed at relatively low cost.

We first discuss the computational complexity without port reduction. Let

³ Note that this is a relatively mild assumption since $\widetilde{\mathbb{A}}_{AA}(\mu)$ shall typically be based on low order port (eigen)modes.

$$m_{\max} \equiv \max_{\text{COM} \in \mathcal{C}_{\text{SYS}}} \sum_{\text{L-P} \in \mathcal{P}_{\text{COM}}} n_{G(\text{L-P}, \text{COM})}, \quad (22)$$

that is, the maximum total number of degrees of freedom associated with any component. The computational costs in the online stage are dominated by $\mathcal{O}(|\mathcal{C}_{\text{SYS}}| m_{\max} (N^3 + Q^2 N^2))$ for the evaluation of the local RB approximations and local RB error bounds; $\mathcal{O}(|\mathcal{C}_{\text{SYS}}| m_{\max}^2 N^2 Q)$ for the assembly of the SCRB system; and $\mathcal{O}(n_{\text{SC}} m_{\max}^2)$ for the solution of the SCRB system and evaluation of $\lambda_{\min}(\mu)$.⁴ In the presence of port reduction, m_{\max} is effectively replaced by the maximum total number of *active* degrees of freedom associated with any component; and n_{SC} is effectively replaced by $n_{\text{A}} < n_{\text{SC}}$ (particularly for r small). Hence the computational cost is significantly reduced. For further details on the computational procedures and costs we refer to Huynh et al. (2011).

3. PORT REPRESENTATION

For each global port $\text{G-P} \in \mathcal{P}_{\text{SYS}}$ we represent the degrees of freedom on the port in terms of functions $\chi_{k, \text{G-P}} \in X^{\mathcal{N}}(\Gamma_{\text{G-P}})$, $1 \leq k \leq n_{\text{G-P}}$. We may truncate this set of functions in the case of port reduction; for properly chosen expansions we will obtain rapid decay in the solution coefficients $\tilde{U}_{k, \text{G-P}}$ such that we may consider only a small subset of the degrees of freedom as *active*, $n_{\text{A}} \ll n_{\text{SC}}$ (and also potentially r small), and hence significantly reduce the online computational cost.⁵ To this end we now introduce a general framework that provides a representation of the degrees of freedom on the ports in terms of eigenfunctions and empirical (snapshot-informed) functions.

We consider any particular $\text{G-P} \in \mathcal{P}_{\text{SYS}}$. We first introduce (a possibly empty) “empirical” space $E_{\text{G-P}}^p \subseteq X^{\mathcal{N}}(\Gamma_{\text{G-P}})$ of dimension $0 \leq p \leq n_{\text{G-P}}$ informed by the solution at G-P for given values of the global parameter. For example, the basis functions for $E_{\text{G-P}}^p$ may be the first p POD modes (Kunisch and Volkwein (2002)) associated with a set of snapshots of the solution at G-P for different parameter values. We also introduce the space $E_{\text{G-P}}^{p, \perp}$ as the $L^2(\Gamma_{\text{G-P}})$ -orthogonal complement of $E_{\text{G-P}}^p$; note that $\dim(E_{\text{G-P}}^{p, \perp}) = n_{\text{G-P}} - p$.

We next introduce the space $X_0^{\mathcal{N}}(\Gamma_{\text{G-P}}) = \{v \in X^{\mathcal{N}}(\Gamma_{\text{G-P}}) : v|_{\partial\Gamma_{\text{G-P}}} = 0\}$ and the following auxiliary problem. Let $s^{\text{Leg}} \in X_0^{\mathcal{N}}(\Gamma_{\text{G-P}})$ satisfy

$$\int_{\Gamma_{\text{G-P}}} \nabla s^{\text{Leg}} \cdot \nabla v = \int_{\Gamma_{\text{G-P}}} v, \quad \forall v \in X_0^{\mathcal{N}}(\Gamma_{\text{G-P}}). \quad (23)$$

We also introduce the function $s^{\text{reg}} \equiv 1$ on $\Gamma_{\text{G-P}}$.

We then introduce the following two eigenproblems *restricted to* $E_{\text{G-P}}^{p, \perp}$. For $* \in \{\text{Leg}, \text{reg}\}$, find $(\chi_{k, \text{G-P}}^*, \lambda_{k, \text{G-P}}^*) \in E_{\text{G-P}}^{p, \perp} \times \mathbb{R}$ such that

$$\int_{\Gamma_{\text{G-P}}} s^* \nabla \chi_{k, \text{G-P}}^* \cdot \nabla v = \lambda_k \int_{\Gamma_{\text{G-P}}} \chi_{k, \text{G-P}}^* v, \quad \forall v \in E_{\text{G-P}}^{p, \perp}, \quad (24)$$

⁴ We here assume that $\tilde{\mathbb{A}}(\mu)$ is a block-sparse matrix with bandwidth m_{\max} and that (13) is solved directly.

⁵ We also choose the $\chi_{k, \text{G-P}}$ such that the dimensions of the RB spaces associated with $\chi_{k, \text{G-P}}$ for larger k are relatively small.

with normalization $\|\chi_{k, \text{G-P}}^*\|_{L^2(\Gamma_{\text{G-P}})} = 1$. Note that in the particular case that $\Gamma_{\text{G-P}} = [-1, 1]$ and $p = 0$, $s^{\text{Leg}}(x) = (1 - x^2)/2$ (for second-order FE elements or in the limit $\mathcal{N} \rightarrow \infty$) and the $\chi_{k, \text{G-P}}^{\text{Leg}}$ are thus equal to the normalized Legendre polynomials.⁶

In the case $p = 0$, these eigenproblems are independent of the connectivity of the components. As a result, the port representation is not specific to any global system configuration and the SCRB framework remains flexible with respect to topological changes. In the case $p > 0$, we lose this online flexibility since $E_{\text{G-P}}^p$ is informed by global snapshots; however we may of course still obtain the solution for different values of the parameters. We expect for a general problem that $* = \text{Leg}$ provides the best approximation since an expansion in Legendre polynomials of a function on $[-1, 1]$ yields exponential coefficient decay assuming only that the solution is smooth. We note that with any choice for p and $*$ in this framework, we may consider ports of rather general shape; in particular, the formulation above is valid for both one-dimensional and two-dimensional ports.

4. NUMERICAL RESULTS

We consider the Poisson problem in the global domain Ω as shown to the right in Figure 1. This global system consists of nine components of the same type as shown to the left in Figure 1; we consider a unity volumetric source in the lower left component and no source term in the remaining eight components. Each component is parametrized by the thermal conductivity in the shaded area (Figure 1, left), which may take values in $[0.1, 10]$; hence $\mathcal{D} = [0.1, 10]^9$. The conductivity outside the shaded area is equal to unity. Each component has four L-Ps (blue) that connect to another component to form twelve G-Ps (red). On the parts of the boundary between the ports we impose homogeneous Neumann boundary conditions; on the L-Ps that do not form G-Ps we impose homogeneous Dirichlet boundary conditions. The underlying FE space $X^{\mathcal{N}}$ is of spectral element type and has dimension $\mathcal{N} = 10560$ (piecewise polynomial order 15); the number of degrees of freedom on each G-P is $n_{\text{G-P}} = 16$ and the size of the Schur complement is $n_{\text{SC}} = 192$. In each case below, the error bounds for the RB approximations to (5) and (6) are typically smaller than 10^{-5} .

We first consider a case in which we do not include any empirical functions, hence $E_{\text{G-P}}^p = 0$. We show in Figure 2 for one value of $\mu \in \mathcal{D}$ the error bound $\widehat{\Delta\mathbb{U}}(\mu)$ and the error $\|\mathbb{U}(\mu) - \tilde{\mathbb{U}}(\mu)\|_2$ as functions of the number of active degrees of freedom for $* = \text{Leg}$ and $* = \text{reg}$ (we use the same number of degrees of freedom on all ports). We note that for this problem both $* = \text{reg}$ and $* = \text{Leg}$ provide exponential convergence; we also note the *effectivities* of the bounds are not too large, $\mathcal{O}(10) - \mathcal{O}(100)$. It is evident that for modest accuracy only a few degrees of freedom on each port are necessary.

We next consider a case with $p = 6$ empirical functions at each G-P obtained as follows. We first sample the global

⁶ The idea of employing eigenfunctions of a geometrically generalized singular eigenproblem as basis functions is also considered in Batcho and Karniadakis (1994) in a very different context.

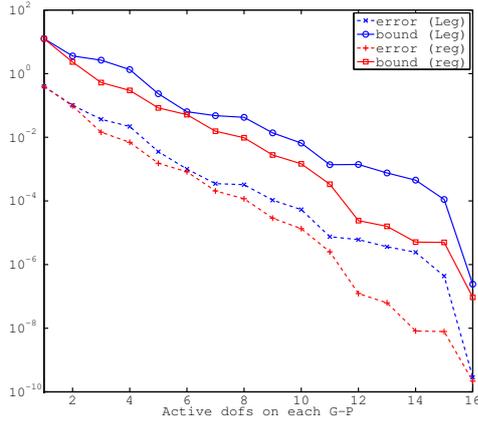


Fig. 2. Error bounds and errors for $p = 0$ with $*$ = reg and $*$ = Leg as functions of the number of active degrees of freedom on each port; we show bounds and errors for $\mu = [0.1, 0.2, 0.4, 0.8, 1.6, 3.2, 6.4, 0.1, 10]^T$.

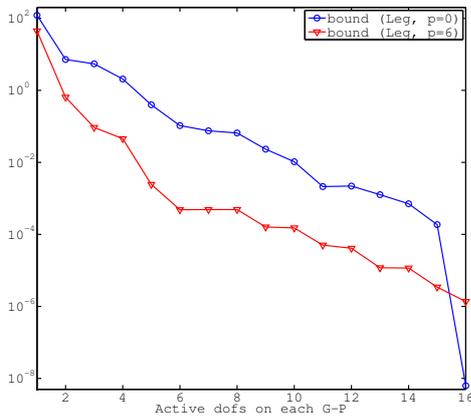


Fig. 3. Error bounds for $*$ = Leg with $p = 0$ and $p = 6$ as functions of the number of active degrees of freedom on each port; we show the maximum bounds over 100 random test parameters.

field for $K \gg p$ parameter values from \mathcal{D} and collect K snapshots at each G-P; we then perform a POD over each of these sets and retain the $p = 6$ first POD modes to obtain a unique space associated with each G-P. The remaining $n_{G-P} - p = 10$ degrees of freedom at each G-P span the $L^2(\Gamma_{G-P})$ -orthogonal complement to this space and are represented as solutions to (24) for $*$ = Leg. We show in Figure 3 the maximum error bound over a set of 100 random test parameters for the case $p = 6$ (red) and $p = 0$ (blue). Clearly the initial empirical modes suffice for a modest error. However, when a very high accuracy is desired or when the global parameter domain can not be sampled adequately due to a large parameter dimension, the orthogonal complement modes ensures convergence.

We finally consider for the case $*$ = Leg and $p = 0$ an online adaptive algorithm for port reduction as follows. We start, for any given $\mu \in \mathcal{D}$, with two degrees of freedom on all G-Ps; we then compute the global error estimator $\widehat{\Delta\bar{U}}_r(\mu)$ as indicated in Remark 1 for $r = 1$; while $\widehat{\Delta\bar{U}}_r(\mu)$

is larger than $\epsilon = 0.1^7$ we compute the mean of the absolute values of the solution coefficients of the last two retained modes at each port⁸ and add a new degree-of-freedom to the G-P at which the maximum value is attained. For 1000 random values of $\mu \in \mathcal{D}$ we obtain with this strategy $\text{mean}(n_A) = 49.4$; hence $n_A \ll n_{SC}$ which leads to significant reduction in online computational cost. The number of degrees of freedom on each port varies from 2 to 7; hence the approach adapts to the complexity of the solution. Note that naive addition of degrees of freedom on all ports at each iteration (no adaptation) results in $\text{mean}(n_A) = 63.0$. We finally compute $\text{mean}(\widehat{\Delta\bar{U}}(\mu)) = 0.13$; hence the error estimator performs well.

Clearly the adaptive algorithm reduces the size of the Schur complement system, but the algorithm is rather expensive. To reduce this cost we may “refine” more than one port at each iteration based on a threshold for the coefficient mean value; we may also add more than one degree-of-freedom at each port. Future work will address optimization of the refinement strategy; 3D problems (with 2D ports), for which n_{G-P} may be large and hence port reduction will play a central role; and dual methods for output error bounds, which will increase the bound effectivity and hence accommodate larger economies in the port representation.

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⁷ The estimator is for absolute error; typically $1 \leq \|\mathbf{U}(\mu)\|_2 \leq 5$.

⁸ For 1D ports we keep a window of two coefficients to account for effects caused by symmetries in the eigenfunctions; for 2D ports a larger window is necessary.

⁹ Available at http://augustine.mit.edu/methodology/papers/HKP_RBcomponents_M2AN.pdf